Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations

Werner Balser

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(continued after the list of symbols)

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Für meine verstorbenen Eltern,

für meine liebe Frau und unsere drei Söhne.

Preface

This book aims at two, essentially different, types of readers: On one hand, there are those who have worked in, or are to some degree familiar with, the section of mathematics that is described here. They may want to have a source of reference to the recent results presented here, replacing my text [21], which is no longer available, but will need little motivation to start using this book. So they may as well skip reading this introduction, or immediately proceed to its second part (p. v) which in some detail describes the content of this book. On the other hand, I expect to attract some readers, perhaps students of colleagues of the first type, who are not familiar with the topic of the book. For those I have written the first part of the introduction, hoping to attract their attention and make them willing to read on.

Some Introductory Examples

What is this book about? If you want an answer in one sentence: It is concerned with formal power series – meaning power series whose radius of convergence is equal to zero, so that at first glance they may appear as rather meaningless objects. I hope that, after reading this book, you may agree with me that these formal power series are fun to work with and really important for describing some, perhaps more theoretical, features of functions solving ordinary or partial differential equations, or difference equations, or perhaps even more general functional equations, which are, however, not discussed in this book.

Do such formal power series occur naturally in applications? Yes, they do, and here are three simple examples:

1. The formal power series $\hat{f}(z) = \sum_0^\infty n!\, z^{n+1}$ formally satisfies the ordinary differential equation (ODE for short)

$$z^2\, x' = x - z. \tag{0.1}$$

But everybody knows how to solve such a simple ODE, so why care about this divergent power series? Yes, that is true! But, given a slightly more complicated ODE, we can no longer *explicitly compute its solutions in closed form.* However, we may still be able to compute solutions in the form of power series. In the simplest case, the ODE may even have a solution that is a polynomial, and such solutions can sometimes be found as follows: *Take a polynomial* $p(z) = \sum_0^m p_n\, z^n$ *with undetermined degree* m *and coefficients* p_n*, insert into the ODE, compare coefficients, and use the resulting equations, which are linear for linear ODE, to compute* m *and* p_n. In many cases, in particular for large m, we may not be able to find the values p_n explicitly. However, we may still succeed in showing that the system of equations for the coefficients has one or several solutions, so that at least the existence of polynomial solutions follows. In other cases, when the ODE does not have polynomial solutions, one can still try to find, or show the existence of, solutions that are "polynomials of infinite degree," meaning power series

$$\hat{f}(z) = \sum_0^\infty f_n\, (z - z_0)^n,$$

with suitably chosen z_0, and f_n to be determined from the ODE. While the approach at first is very much analogous to that for polynomial solutions, two new problems arise: For one thing, we will get a system of infinitely many equations in infinitely many unknowns, namely, the coefficients f_n; and secondly, we are left with the problem of determining the radius of convergence of the power series. The first problem, in many cases, turns out to be relatively harmless, because the system of equations usually can be made to have the form of a *recursion*: Given the coefficients $f_0, \ldots, f_n$, we can then compute the next coefficient f_{n+1}. In our example (0.1), trying to compute a power series solution $\hat{f}(z)$, with $z_0 = 0$, immediately leads to the identities $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = n\, f_n$, $n \geq 1$. Even to find the radius of convergence of the power series may be done, but as the above example shows, *it may turn out to be equal to zero!*

2. Consider the *difference equation*

$$x(z+1) = (1 - a\, z^{-2})\, x(z).$$

After some elementary calculations, one can show that this difference equation has a unique solution of the form $\hat{f}(z) = 1 + \sum_1^\infty f_n\, z^{-n}$, which is a power series in $1/z$. The coefficients can be uniquely computed from the recursion obtained from the difference equation, and they grow, roughly speaking, like $n!$ so that, as in the previous case, the radius of convergence of the power series is equal to zero. Again, this example is so simple that one can explicitly compute its solutions in terms of Gamma functions. But only slightly more complicated difference equations cannot be solved in closed form, while they still have solutions in terms of formal power series.

3. Consider the following problem for the heat equation:

$$u_t = u_{xx}, \quad u(0,x) = \varphi(x),$$

with a function φ that we assume holomorphic in some region G. This problem has a unique solution $u(t,x) = \sum_0^\infty u_n(x)\, t^n$, with coefficients given by

$$u_n(x) = \frac{\varphi^{(2n)}(x)}{n!}, \qquad n \ge 0.$$

This is a power series in the variable t, whose coefficients are functions of x that are holomorphic in G. As can be seen from Cauchy's Integral Formula, the coefficients $u_n(x)$, for fixed $x \in G$, in general grow like $n!$ so that the power series has radius of convergence equal to zero.

So formal power series do occur naturally, but what are they good for? Well, this is exactly what this book is about. In fact, it presents two different but intimately related aspects of formal power series:

For one thing, the very general theory of *asymptotic power series expansions* studies certain functions f that are holomorphic in a sector S but singular at its vertex, and have a certain *asymptotic behavior* as the variable approaches this vertex. One way of describing this behavior is by saying that the nth derivative of the function approaches a limit f_n as the variable z, inside of S, tends toward the vertex z_0 of the sector. As we shall see, this is equivalent to saying that the function, in some sense, is *infinitely often differentiable at z_0, without being holomorphic there, because the limit of the quotient of differences will only exist once we stay inside of the sector.* The values f_n may be regarded as the coefficients of *Taylor's series* of f, but this series may not converge, and even when it does, it *may not converge toward the function f*. Perhaps the simplest example of this kind is the function $f(z) = \mathrm{e}^{-1/z}$, whose derivatives all tend to $f_n = 0$ whenever z tends toward the origin in the right half-plane. This also shows that, unlike for functions that are holomorphic at z_0, this Taylor series alone does not determine the function f. In fact, given any sector S, *every* formal power series $\hat{f}$ arises as an asymptotic expansion of *some* f that is holomorphic in S, but this f *never is uniquely determined by* $\hat{f}$, so that in particular the value of the function at a given point $z \neq z_0$ *in general cannot be computed from the asymptotic power series.* In this book, the theory of asymptotic power series expansions is presented, not only for the case when the coefficients are numbers, but also for series whose coefficients are in a given Banach space. This generalization is strongly motivated by the third of the above examples.

While general formal power series *do not determine one function*, some of them, especially the ones arising as solutions of ODE, are *almost as well-behaved as convergent ones*: One can, more or less explicitly, *compute some function* f from the divergent power series $\hat{f}$, which in a certain sector is asymptotic to $\hat{f}$. In addition, this function f has other very natural properties; e.g., it satisfies the same ODE as $\hat{f}$. This theory of *summability* of formal power series has been developed very recently and is the main reason why this book was written.

If you want to have a simple example of how to compute a function from a divergent power series, take $\hat{f}(z) = \sum_0^\infty f_n z^n$, assuming that $|f_n| \le n!$ for $n \ge 0$. Dividing the coefficients by $n!$ we obtain a new series converging at least for $|z| < 1$. Let $g(z)$ denote its sum, so g is holomorphic in the unit disc. Now the general idea is to define the integral

$$f(z) = z^{-1} \int_0^\infty g(u)\, \mathrm{e}^{-u/z}\, du \tag{0.2}$$

as the sum of the series $\hat{f}$. One reason for this to be a suitable definition is the fact that if we replace the function g by its power series and integrate termwise (which is illegal in general), then we end up with $\hat{f}(z)$. While this motivation may appear relatively weak, it will become clear later that this nonetheless is an excellent definition for a function f deserving the title *sum of* $\hat{f}$ – *except that the integral* (0.2) *may not make sense for one of the following two reasons:* The function g is holomorphic in the unit disc but may be undefined for values u with[1] $u \ge 1$, making the integral entirely meaningless. But even if we assume that g can be holomorphically continued along the positive real axis, its rate of growth at infinity may be such that the integral diverges. So you see that there are some reasons that keep us from getting a meaningful sum for $\hat{f}$ in this simple fashion, and therefore we shall have to consider more complicated ways of *summing* formal power series. Here we shall present a summation process, called *multisummability*, that can handle every formal power series which solves an ODE, but is still not general enough for solutions of certain difference equations or partial differential equations. *Jean Ecalle*, the founder of the theory of multisummability, has also outlined some more general

[1] Observe that such an inequality should be understood as saying: Here, the number u must be *real* and at least 1.

summation methods suitable for difference equations, but we shall not be concerned with these here.

Content of this Book

This book attempts to present the theory of linear ordinary differential equations in the complex domain from the new perspective of multisummability. It also briefly describes recent efforts on developing an analogous theory for *nonlinear systems, systems of difference equations, partial differential equations, and singular perturbation problems.* While the case of linear systems may be said to be very well understood by now, much more needs to be done in the other cases.

The material of the book is organized as follows: The first two chapters contain entirely classical results on the structure of solutions near regular, resp.[2] regular-singular, points. They are included here mainly for the sake of completeness, since none of the problems that the theory of multisummability is concerned with arise in these cases. A reader with some background on ODE in the complex domain may very well skip these and immediately advance to Chapter 3, where we begin discussing the *local theory of systems near an irregular singularity.* Classically, this theory starts with showing existence of formal fundamental solutions, which in our terminology will turn out to be *multisummable, but not k-summable, for any $k > 0$.* So in a way, these classical formal fundamental solutions are relatively complicated objects. Therefore, we will in Chapter 3 introduce a different kind of what we shall call *highest level formal fundamental solutions*, which have much better theoretical properties, although they are somewhat harder to compute. In the following chapters we then present the theory of asymptotic power series with special emphasis on Gevrey asymptotics and k-summability.

In contrast to the presentation in [21], we here treat *power series with coefficients in a Banach space.* The motivation for this general approach lies in applications to PDE and singular perturbation problems that shall be discussed briefly later. A reader who is not interested in this general setting may concentrate on series with coefficients in the complex number field, but the general case really is not much more difficult.

In Chapters 8 and 9 we then return to the theory of ODE and discuss the *Stokes phenomenon* of highest level. Here it is best seen that the approach we take here, relying on highest level formal fundamental solutions, gives a far better insight into the structure of the Stokes phenomenon, because it avoids mixing the phenomena occurring on different levels. Nonetheless, we then present the theory of multisummability in the following chapters and indicate that the classical formal fundamental solutions are indeed multisummable. The remaining chapters of the book are devoted to related but

[2]Short for “respectively.”

different problems such as *Birkhoff's reduction problem* or applications of the theory of multisummability to *difference equations, partial differential equations, or singular perturbation problems.* Several appendices provide the results from other areas of mathematics required in the book; in particular some well-known theorems from the theory of complex variables are presented in the more general setting of functions with values in a Banach space.

The book should be readable for students and scientists with some background in matrix theory and complex analysis, but I have attempted to include all the (nonelementary) results from these areas in the appendices. A reader who is mainly interested in the asymptotic theory and/or multisummability may leave out the beginning chapters and start reading with Chapters 4 through 7, and then go on to Chapter 10 – these are pretty much independent of the others in between and may be a good basis for a course on the subject of asymptotic power series, although the remaining ones may provide an excellent motivation for such a general theory to be developed.

Personal Remarks

Some personal remarks may be justified here: In fall of 1970, I came to the newly founded University of Ulm to work under the direction of *Alexander Peyerimhoff* in *summability theory.* About 1975 I switched fields and, jointly with *W. B. Jurkat* and *D. A. Lutz*, began my studies in the very classical, yet still highly active, field of *systems of ordinary linear differential equations whose coefficients are meromorphic functions of a complex variable* (for short: meromorphic systems of ODE). This field has occupied most of my (mathematical) energies, until almost twenty years later when I took up summability again to apply its techniques to the *divergent power series* that arise as *formal solutions* of meromorphic ODE. In this book, I have made an effort to represent the classical theory of meromorphic systems of ODE in the new light shed upon it by the recent achievements in the theory of summability of formal power series.

After more than twenty years of research, I have become highly addicted to this field. I like it so much because it gives us a splendid opportunity to obtain significant results using standard techniques from the theory of complex variables, together with some matrix algebra and other classical areas of analysis, such as summability theory, and I hope that this book may infect others with the same enthusiasm for this fascinating area of mathematics. While one may also achieve useful results using more sophisticated tools borrowed from advanced algebra, or functional analysis, such will not be required to understand the content of this book.

I should like to make the following acknowledgments: I am indebted to the group of colleagues at Grenoble University, especially *J. DellaDora, F. Jung*, and *M. Barkatou* and his students. During my appointment as *Pro-*

fesseur Invité in September 1997 and February 1998, they introduced me to the realm of computer algebra and helped me prepare the corresponding section, and in addition created a perfect environment for writing a large portion of the book while I was there. In March 1998, while I was similarly visiting Lille University, *Anne Duval* made me appreciate the very recent progress on application of multisummability to the theory of *difference equations*, for which I am grateful as well. I would also like to thank many other colleagues for support in collecting the numerous references I added, and for introducing me to related, yet different, applications of multisummability on formal solutions of partial differential equations and singular perturbation problems. Last, but not least, I owe thanks to my two teachers at Ulm University, *Peyerimhoff* (who died all too suddenly in 1996) and *Jurkat*, who were not actively involved in writing, but from whom I acquired the mathematics, as well as the necessary stamina, to complete this book.

Ulm, Germany, 1999 Werner Balser

Contents

1

Basic Properties of Solutions

In this first chapter, we discuss some basic properties of linear systems of ordinary differential equations having a coefficient matrix whose entries are holomorphic functions in some region G. A reader who is familiar with the theory of systems whose coefficient matrix is constant, or consists of continuous functions on a real interval, will see that all of what we say here for the case of a *simply connected region* G, i.e., a region "without holes," is quite analogous to the real-variable situation, but we shall discover a new phenomenon in case of multiply connected G. While for simply connected regions solutions always are holomorphic *in the whole region* G, this will no longer be true for multiply connected ones: Solutions will be *locally holomorphic*, i.e., holomorphic on every disc contained in G. Globally, however, they will in general be *multivalued* functions that should best be considered on some Riemann surface over G. As an example, observe that $x(z) = \sqrt{z(1-z)}$ is a solution of the equation

$$x' = \frac{1-2z}{2z\,(1-z)}\, x, \quad z \in G = \mathbb{C} \setminus \{0,1\};$$

this solution has branch-points at $z = 1$ and the origin. Luckily, we shall have no need to study this *monodromy behavior* of solutions for a general multiply connected region. Instead, it will be sufficient to consider the simplest type of such regions, namely *punctured discs*. Assuming for simplicity that we have a punctured disc about the origin, the corresponding Riemann surface – or to be exact, the universal covering surface – is the Riemann surface of the (natural) logarithm. We require the reader to have some intuitive understanding of this concept, but we shall also discuss this surface on p. 226 in the Appendix.

Most of the time we shall restrict ourselves to systems of first-order linear equations. Since every νth order equation can be rewritten as a system (see Exercise 5 on p. 4), our results carry over to such equations as well. However, in some circumstances scalar equations are easier to handle than systems. So for practical purposes, such as computing power series solutions, we do not recommend to turn a given scalar equation into a system, but instead one should work with the scalar equation directly.

Many books on ordinary differential equations contain at least a chapter or two dealing with ODE in the complex plane. Aside from the books of Sibuya and Wasow, already mentioned in the introduction, we list the following more recent books in chronological order: *Ince* [138], *Bieberbach* [52], *Schäfke and Schmidt* [236], and *Hille* [120].

1.1 Simply Connected Regions

Throughout this chapter, we consider a system of the form

$$x' = A(z)\, x, \qquad z \in G, \tag{1.1}$$

where $A(z) = [a_{kj}(z)]$ denotes a $\nu \times \nu$ matrix whose entries are *holomorphic functions in some fixed region* $G \subset \mathbb{C}$, which we here assume to be *simply connected*. It is notationally convenient to think of such a matrix $A(z)$ as a *holomorphic matrix-valued function* in G.

Since we know from the theory of functions of a complex variable that such functions, if (once) differentiable in an open set, are automatically holomorphic there, it is obvious that solutions $x(z)$ of (1.1) are always vector-valued holomorphic functions. However, it is not clear off-hand that a solution always is holomorphic in *all of* the region G, but we shall prove this here. To begin, we show the following weaker result, which holds for arbitrary regions G.

Lemma 1 *Let a system* (1.1)*, with* $A(z)$ *holomorphic in a region* $G \subset \mathbb{C}$*, be given. Then for every* $z_0 \in G$ *and every* $x_0 \in \mathbb{C}^\nu$*, there exists a unique vector-valued function* $x(z)$*, holomorphic in the largest disc* $D = D(z_0, \rho) = \{z : |z - z_0| < \rho\}$ *contained in* G*, such that*

$$x'(z) = A(z)\, x(z), \quad z \in D, \quad x(z_0) = x_0.$$

Hence we may say for short that every initial value problem has a unique solution that is holomorphic near z_0*.*

Proof: Assume for the moment that we were given a solution $x(z)$, holomorphic in D. Then the coordinates of the vector function $x(z)$ all can be expanded into power series about z_0, with a radius of convergence at least ρ. Combining these series into a vector, we can expand $x(z)$ into a *vector power series*[1]

$$x(z) = \sum_0^\infty x_n \, (z - z_0)^n, \quad |z - z_0| < \rho, \tag{1.2}$$

where $x_n \in \mathbb{C}^{\,\nu}$ are the *coefficient vectors*. Likewise, we can expand the coefficient matrix $A(z)$ into a *matrix power series*

$$A(z) = \sum_0^\infty A_n \, (z - z_0)^n, \quad |z - z_0| < \rho, \tag{1.3}$$

with *coefficient matrices* $A_n \in \mathbb{C}^{\,\nu\times\nu}$. Inserting these expansions into the system (1.1) and comparing coefficients leads to the identities

$$(n+1)\, x_{n+1} = \sum_{m=0}^{n} A_{n-m} \, x_m, \quad n \ge 0. \tag{1.4}$$

Hence, given x_0, we can recursively compute x_n for $n \ge 1$ from (1.4), which proves the uniqueness of the solution. To show existence, it remains to check whether the *formal power series solution* of our initial value problem, resulting from (1.4), converges for $|z - z_0| < \rho$. To do this, note that convergence of (1.3) implies $\|A_n\| \le c\, K^n$ for every constant $K > 1/\rho$ and sufficiently large $c > 0$, depending on K. Hence, equation (1.4) implies $(n+1)\, \|x_{n+1}\| \le c \sum_{m=0}^{n} K^{n-m} \, \|x_m\|$, $n \ge 0$. Defining a *majorizing sequence* (c_n) by $c_0 = \|x_0\|$, resp. $(n+1)\, c_{n+1} = c \sum_{m=0}^{n} K^{n-m} \, c_m$, $n \ge 0$, we conclude by induction that $\|x_n\| \le c_n$, for $n \ge 0$. The power series $f(z) = \sum_0^\infty c_n \, z^n$ can be easily checked to formally satisfy the linear ODE $y' = c\,(1 - K z)^{-1}\, y$. This equation has the solution $y(z) = c_0\,(1 - K z)^{-c/K}$, which is holomorphic in the disc $|z| < 1/K$. Expanding this function into its power series about the origin, inserting into the ODE and comparing coefficients, one checks that the coefficients satisfy the same recursion relation as the c_n, hence are, in fact, equal to the numbers c_n. This proves that the radius of convergence of (1.2) is at least $1/K$, and since K was arbitrary except for $K > 1/\rho$, the proof is completed.[2] □

Using the above *local result* together with the monodromy theorem on p. 225 in the Appendix, it is now easy to show the following *global version* of the same result:

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[1] Observe that whenever we write $|z - z_0| < \rho$, or a similar condition on z, we wish to state that the corresponding formula holds, and here in particular the series converges, for such z.

[2] An alternative proof for convergence of $f(z)$ is as follows: Show $(n+1)\, c_{n+1} = c \sum_{m=0}^{n} K^{n-m} \, c_m = (c + K n)\, c_n$, hence the quotient test implies convergence of $f(z)$ for $|z| < 1/K$. While this argument is simpler, it depends on the structure of the recursion relation for (c_n) and fails in more general cases; see, e.g., the proof of Lemma 2 (p. 28).

Theorem 1 *Let a system* (1.1), *with* $A(z)$ *holomorphic in a simply connected region* $G \subset \mathbb{C}$, *be given. Then for every* $z_0 \in G$ *and every* $x_0 \in \mathbb{C}^\nu$, *there exists a unique vector-valued function* $x(z)$, *holomorphic in* G, *such that*

$$x'(z) = A(z)\,x(z), \quad z \in G, \quad x(z_0) = x_0. \tag{1.5}$$

Proof: Given any path γ in G originating at z_0, we may cover the path with finitely many circles in G, such that when proceeding along γ, each circle contains the midpoint of the next one. Applying Lemma 1 successively to each circle, one can show that the unique local solution of the initial value problem can be holomorphically continued along the path γ. Since any two paths in a simply connected region are always homotopic, the monodromy theorem mentioned above completes the proof. □

Exercises: In the following exercises, let G be a simply connected region, and $A(z)$ a matrix-valued function, holomorphic in G.

1. Show that the set of all solutions of (1.1) is a vector space over $\mathbb{C}$. For its dimension, see Theorem 2 (p. 6).

2. Give a different proof of Lemma 1, analogous to that of Picard-Lindelöf's theorem in the real variable case.

3. Check that the proof of the previous exercise, with minor modifications, may be used to prove the same result with the disc D replaced by the largest subregion of G, which is star-shaped with respect to z_0.

4. Use Riemann's mapping theorem [1] and Lemma 1 to obtain another proof of Theorem 1.

5. Consider the following νth-order linear ODE:

$$y^{(\nu)} - a_1(z)\,y^{(\nu-1)} - \ldots - a_\nu(z)\,y = 0, \tag{1.6}$$

where $a_k(z)$ are (scalar) holomorphic functions in G. Introducing the matrix

$$A(z) = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ a_\nu(z) & a_{\nu-1}(z) & \ldots & a_2(z) & a_1(z) \end{bmatrix},$$

conclude from Theorem 1 that all solutions of (1.6) are holomorphic in G. A matrix of the above form will be called a *companion matrix* corresponding to the row vector $a(z) = (a_\nu(z), \ldots, a_1(z))$.

6. Consider the following second-order ODE, usually called *Legendre's equation*:
$$[(1-z^2)\,x'\,]' + \mu\,x = 0, \qquad \mu \in \mathbb{C}. \tag{1.7}$$
   (a) Write (1.7) in the form (1.6) and determine where the coefficients $a_k(z)$ are holomorphic resp. singular.
   (b) Insert $x(z) = \sum_0^\infty x_n\, z^n$ into (1.7), compare coefficients, and find the resulting recursion for the x_n. Without explicitly finding the coefficients, find the radius of convergence of the power series.
   (c) For $\mu = m\,(m+1)$, $m \in \mathbb{N}_0$, show that (1.7) has a solution that is a polynomial of degree m. These polynomials are called *Legendre's polynomials.*
   (d) Verify that the values $\mu = m\,(m+1)$ are the only ones for which a nontrivial polynomial solution can exist.

7. For a system (1.1), choose an arbitrary row vector $t_0(z)$ that is holomorphic in G, and define inductively
$$t_{k+1}(z) = t_k'(z) + t_k(z)\,A(z), \quad z \in G,\ k \ge 0. \tag{1.8}$$
   Let $T(z)$ be the matrix with rows $t_0(z), \ldots, t_{\nu-1}(z)$; hence $T(z)$ is holomorphic in G.
   (a) Show that one can choose $t_0(z)$ so that $\det T(z)$ does not vanish identically on G. If this is so, the vector $t_0(z)$ will be called a *cyclic vector* for $A(z)$. More precisely, for arbitrary $z_0 \in G$ and sufficiently small $\rho > 0$, show the existence of a cyclic vector for which $\det T(z) \neq 0$ on $D(z_0, \rho)$.
   (b) For $T(z)$ as above, define $b(z)$ by $t_\nu(z) = b(z)\,T(z)$ on D, and let $B(z)$ be the companion matrix corresponding to $b(z)$. Conclude for $\tilde{x}(z) = T(z)\,x(z)$ that $x(z)$ solves (1.1) (on G) if and only if $\tilde{x}'(z) = B(z)\,\tilde{x}(z)$, $z \in D$. Compare this to the previous exercise.

1.2 Fundamental Solutions

As is common in the real theory of linear systems of ODE, we say that a $\nu \times \nu$ matrix-valued function $X(z)$ is a *fundamental solution* of (1.1), if all columns are solutions of (1.1), so that in particular $X(z)$ is holomorphic in G, and if in addition the determinant of $X(z)$ is nonzero; note that according to the following proposition $\det X(z_0) \neq 0$ for some $z_0 \in G$ already implies $\det X(z) \neq 0$ for every $z \in G$:

Proposition 1 (Wronski's Identity) *Consider a holomorphic matrix-valued function $X(z)$ satisfying $X'(z) = A(z)\,X(z)$ for $z \in G$, and let $w(z) = \det X(z)$ and $a(z) = \operatorname{trace} A(z)$. Then*

$$w(z) = w(z_0)\ \exp\left[\int_{z_0}^{z} a(u)\,du\right], \quad z \in G,$$

for arbitrary $z_0 \in G$. Hence in particular, either $w(z) \equiv 0$ or $w(z) \neq 0$ for every $z \in G$.

Proof: The definition of determinants shows $w'(z) = \sum_{k=1}^{\nu} w_k(z)$, where $w_k(z)$ is the determinant of the matrix obtained from $X(z)$ by differentiating its kth row and leaving the others. If $r_k(z)$ denotes the kth row of $X(z)$, then $X'(z) = A(z)\,X(z)$ implies $r_k'(z) = \sum_{j=1}^{\nu} a_{kj}(z)\,r_j(z)$. Using this and observing that the determinant of matrices with two equal rows vanishes, we obtain $w_k(z) = a_{kk}(z)\,w(z)$, hence

$$w'(z) = a(z)\,w(z).$$

Solving this differential equation then completes the proof. □

Existence of fundamental solutions is clear: For fixed $z_0 \in G$, take the unique solution of the initial value problem (1.5) with $x_0 = e_k$, the kth unit vector, for $1 \leq k \leq \nu$. Combining these vectors as columns of a matrix $X(z)$, we see that $\det X(z_0) = 1$; hence $X(z)$ is a fundamental solution. The significance of fundamental solutions is that, as for the real case, *every solution* of (1.1) is a linear combination of the columns of a fundamental one:

Theorem 2 *Suppose that $X(z)$ is a fundamental solution of* (1.1), *and let $x(z)$ be the unique solution of the initial value problem* (1.5). *For $c = X^{-1}(z_0)\,x_0$, we then have*

$$x(z) = X(z)\,c, \quad z \in G.$$

Thus, the $\mathbb{C}$-vector space of all solutions of (1.1) *is of dimension ν, and the columns of $X(z)$ are a basis.*

Proof: First observe that $X(z)\,c$, for any $c \in \mathbb{C}^{\nu}$, is a solution of (1.1). Defining c as in the theorem then leads to a solution satisfying the initial value condition, so the proof is completed. □

In principle, the *computation of the power series expansion* of a fundamental solution of (1.1) presents no new problem: For $z_0 \in G$ and $x_0 = e_k$, $1 \leq k \leq \nu$, compute the power series expansion of the solution of (1.5) as in the proof of Lemma 1, thus obtaining a power series representation of a fundamental solution near z_0 of the form $X(z) = \sum_0^\infty X_n\,(z - z_0)^n$, with $X_0 = I$, and

$$(n+1)\,X_{n+1} = \sum_{m=0}^{n} A_{n-m}\,X_m, \quad n \ge 0. \tag{1.9}$$

Re-expanding this power series, one then can holomorphically continue the fundamental solution into all of G. However, in practically all cases, it will not be possible to compute *all* coefficients X_n, and even if we succeeded, the process of holomorphic continuation would be extremely tedious, if not impossible. So on one hand, *the recursion equations* (1.9) *contain all the information about the global behavior of the corresponding fundamental solution, but to extract such information explicitly must be considered highly non-trivial.* Much of what follows will be about other ways of representing fundamental solutions, which then allow us to learn more about their behavior, e.g, near a boundary point of the region G.

Exercises: Throughout the following exercises, let a system (1.1) on a simply connected region G be arbitrarily given.

1. Let $X(z)$ be a fundamental solution of (1.1). Show that $\tilde X(z)$ is another fundamental solution of (1.1) if and only if $\tilde X(z) = X(z)\,C$ for some constant invertible $\nu \times \nu$ matrix C.

2. For $A(z) = A\,z^k$, with $k \in \mathbb{N}_0$ and $A \in \mathbb{C}^{\,\nu\times\nu}$ (and $G = \mathbb{C}$), show that $X(z) = \exp[A\,z^{k+1}/(k+1)]$ is a fundamental solution of (1.1).

3. For $B(z)$ commuting with $A(z)$ and $B'(z) = A(z),\ z \in G$, show that $X(z) = \exp[B(z)],\ z \in G$, is a fundamental solution of (1.1).

4. For $\nu = 2$ and $a_{11}(z) = a_{22}(z) \equiv 0,\ a_{21}(z) \equiv 1,\ a_{12}(z) = z$, show that no $B(z)$ exists so that $B(z)$ commutes with $A(z)$ and $B'(z) = A(z),\ z \in G$, for whatever region G.

5. Let $X(z)$ be a holomorphic matrix, with $\det X(z) \ne 0$ for every $z \in G$. Find $A(z)$ such that $X(z)$ is a fundamental solution of (1.1).

6. Show that $X(z)$ is a fundamental solution of (1.1) if and only if $[X^{-1}(z)]^T$ is one for $\tilde x' = -A^T(z)\,\tilde x$.

7. Let $x_1(z), \ldots, x_\mu(z),\ \mu < \nu$, be solutions of (1.1). Show that the rank of the matrix $X(z) = [x_1(z), \ldots, x_\mu(z)]$ is constant, for $z \in G$.

8. Let $x_1(z), \ldots, x_\mu(z),\ \mu < \nu$, be linearly independent solutions of (1.1). Show existence of holomorphic vector functions $x_{\mu+1}(z), \ldots, x_\nu(z)$, $z \in G$, so that for some, possibly small, subregion $\tilde G \subset G$ we have $\det[x_1(z), \ldots, x_\nu(z)] \ne 0$ on $\tilde G$. For $T(z) = [x_1(z), \ldots x_\nu(z)]$, set $x = T(z)\,\tilde x$ and conclude that $x(z)$ satisfies (1.1) if and only if $\tilde x$ solves

$$\tilde x' = \tilde A(z)\,\tilde x, \qquad z \in \tilde G,$$

for $\tilde{A}(z) = T^{-1}(z)\,[A(z)\,T(z) - T'(z)]$. Show that the first μ columns of $\tilde{A}(z)$ vanish identically. Compare this to Exercise 4 on p. 14.

1.3 Systems in General Regions

We now consider a system (1.1) in a general region G. Given a fundamental solution $X(z)$, defined near some point $z_0 \in G$, we can holomorphically continue $X(z)$ along any path γ in G beginning at z_0 and ending, say, at z_1. Clearly, this process of holomorphic continuation produces a solution of (1.1) near the point z_1. Since the path can be split into finitely many pieces, such that each of them is contained in a simply connected subregion of G to which the results of the previous section apply, $\det X(z)$ cannot vanish. Thus, $X(z)$ remains fundamental during holomorphic continuation. According to the monodromy theorem, for *a different path from z_0 to z_1 the resulting fundamental solution near z_1 will be the same provided the two paths are homotopic.* In particular, if γ is a Jordan curve whose interior region belongs to G, so that γ does not wind around exterior points of G, then holomorphic continuation of $X(z)$ along γ reproduces the same fundamental solution that we started with. However, if the interior of γ contains points from the complement of G, then simple examples in the exercises below show that in general we shall obtain a different one. Hence Theorem 1 (p. 4) fails for multiply connected G, since holomorphic continuation may not lead to a fundamental solution that is holomorphic in G, but rather on a *Riemann surface* associated with G. We shall not go into details about this, but will be content with the following result for a *punctured disc* $R(z_0, \rho) = \{z : \ 0 < |z - z_0| < \rho\}$, or slightly more general, an arbitrary *ring* $R = \{z : \ \rho_1 < |z - z_0| < \rho\}$, $0 \le \rho_1 < \rho$:

Proposition 2 *Let a system* (1.1), *with $G = R$ as above, be given. Let $X(z)$ denote an arbitrary fundamental solution of* (1.1) *in a disc $D = D(z_1, \tilde{\rho}) \subset R$, $\tilde{\rho} > 0$. Then there exists a matrix $M \in \mathbb{C}^{\nu\times\nu}$ such that, for a fixed but arbitrary choice of the branch of $(z - z_0)^M = \exp[M \log(z - z_0)]$ in D, the matrix*

$$S(z) = X(z)\ (z - z_0)^{-M}$$

is single-valued in R.

Proof: Continuation of $X(z)$ along the circle $|z - z_0| = |z_1 - z_0|$ in the positive sense will produce a fundamental solution, say, $\tilde{X}(z)$, of (1.1) in D. According to Exercise 1 on p. 7, there exists an invertible matrix $C \in \mathbb{C}^{\nu\times\nu}$ so that $\tilde{X}(z) = X(z)\,C$ for $z \in D$. Choose M so that $C = \exp[2\pi i\, M]$, e.g., $2\pi i\, M = \log M$. Then, continuation of $(z - z_0)^M$ along the same circle leads to $\exp[M(\log(z - z_0) + 2\pi i)] = (z - z_0)^M C$, which completes the proof. □

While the matrix C occurring in the above proof is uniquely defined by the fundamental solution $X(z)$, the matrix M is not! We call any such M a *monodromy matrix* for $X(z)$. The unique matrix C is sometimes called *the monodromy factor* for $X(z)$. Observe that M can be *any matrix* with $\exp[2\pi i\, M] = C$. So, in general, $2\pi i\, M$ may have eigenvalues differing by nonzero integers, *and then* $2\pi i\, M$ *is not a branch for the matrix function* $\log C$.

For $G = R$, it is convenient to think of solutions of (1.1) as defined on the *Riemann surface of the natural logarithm* of $z - z_0$, as described on p. 226 in the Appendix. This surface can best be visualized as a spiraling staircase with infinitely many levels in both directions. For simplicity, take $z_0 = 0$, then traversing a circle about the origin in the positive, i.e., counterclockwise, direction, will not take us back to the same point, as it would in the complex plane, but to the one on the next level, directly above the point where we started. Thus, while complex numbers $z_k = r\, \mathrm{e}^{i\varphi_k}$, $r > 0$, are the same once their arguments φ_k differ by integer multiples of 2π, *the corresponding points on the Riemann surface are different.* So strictly speaking, instead of complex numbers $z = r\, \mathrm{e}^{2\pi i\varphi}$, we deal with *pairs* (r, φ). On this surface, the matrix $z^M = \exp[M \log z]$ becomes a single-valued holomorphic function by interpreting $\log z = \log r + i\varphi$.

The above proposition shows that, once we have a monodromy matrix M, we completely understand the branching behavior of the corresponding fundamental solution $X(z)$. It pays to work out the general form of z^M for $M = J$ in Jordan form, in order to understand the various cases that can occur for the branching behavior of $X(z)$.

The computation of monodromy matrices and/or their eigenvalues is a major task in many applications. In principle, it should be possible to find them by first computing a fundamental solution $X(z)$ by means of the recursions (1.9), and then iteratively re-expanding the resulting power series to obtain the analytic continuation. In reality there is little hope of effectively doing this. So it will be useful to obtain other representations for fundamental solutions providing more direct ways for finding monodromy matrices. For singularities of the first kind, which are discussed in the next chapter, this can always be done, while for other cases this problem will prove much more complicated.

Exercises: Throughout these exercises, let $M \in \mathbb{C}^{\nu\times\nu}$.

1. Verify that $X(z) = z^M = \exp[M \log z]$ is a fundamental solution of $x' = z^{-1} M\, x$ near, e.g., $z_0 = 1$, if we select any branch for the multivalued function $\log z$, e.g., its *principal value*, which is real-valued along the positive real axis.

2. Verify that $X(z) = z^M$ in general cannot be holomorphically continued (as a single-valued holomorphic function) into all of $R(0, \infty)$.

3. Verify that M is a monodromy matrix for $X(z) = z^M$.

4. Let $M_k \in \mathbb{C}^{\nu\times\nu}$, $1 \le k \le \mu$, be such that they all commute with one another, let $A(z) = \sum_{k=1}^{\mu}(z - z_k)^{-1} M_k$, with all distinct $z_k \in \mathbb{C}$, and $G = \mathbb{C} \setminus \{z_1, \ldots, z_\mu\}$. For each k, $1 \le k \le \mu$, and ρ sufficiently small, show the existence of a fundamental solution of (1.1) in $R(z_k, \rho)$ having monodromy matrix M_k.

5. For M as above and any matrix-valued $S(z)$, holomorphic and single-valued with $\det S(z) \neq 0$ in $z \in R(0, \rho)$, for some $\rho > 0$, find $A(z)$ so that $X(z) = S(z)\, z^M$ is a fundamental solution of (1.1).

6. For $G = R(0, \rho)$, $\rho > 0$, show that monodromy factors for different fundamental solutions of (1.1) are similar matrices. Show that the eigenvalues of corresponding monodromy matrices always are congruent modulo one in the following sense: If M_1, M_2 are monodromy matrices for fundamental solutions $X_1(z), X_2(z)$ of (1.1), then for every eigenvalue μ of M_1 there exists $k \in \mathbb{Z}$ so that $k+\mu$ is an eigenvalue of M_2.

7. Under the assumptions of the previous exercise, let M_1 be a monodromy matrix for some fundamental solution. Show that one *can choose* a monodromy matrix M_2 for another fundamental solution so that both are similar. Verify that, for a given fundamental solution, one can always choose a unique monodromy matrix that has eigenvalues with real parts in the half-open interval $[0, 1)$.

8. Under the assumptions of the previous exercises, show the existence of at least one solution vector of the form $x(z) = s(z)\, z^\mu$, with $\mu \in \mathbb{C}$ and $s(z)$ a single-valued vector function in G.

9. Consider the scalar ODE (1.6) for $a_k(z)$ holomorphic in $G = R(0, \rho)$, $\rho > 0$. Show that (1.6) has at least one solution of the form $y(z) = s(z)\, z^\mu$, with $\mu \in \mathbb{C}$ and a scalar single-valued function $s(z)$, $z \in G$.

1.4 Inhomogeneous Systems

We now return to a simply connected region G, but will consider an *inhomogeneous* system

$$x' = A(z)\, x + b(z), \quad z \in G, \tag{1.10}$$

where $b(z)$ is a vector-valued holomorphic function on G. We then refer to (1.1) as the *corresponding homogeneous system.* As in the real variable case, we can solve (1.10) as soon as a fundamental solution of the corresponding homogeneous system (1.1) is known:

Theorem 3 (Variation of Constants Formula) *For a simply connected region G, and $A(z)$, $b(z)$ holomorphic in G, all solutions of* (1.10) *are holomorphic in G and given by the formula*

$$x(z) = X(z)\left(c + \int_{z_0}^{z} X^{-1}(u)\, b(u)\, du\right), \quad z \in G, \tag{1.11}$$

where $z_0 \in \mathbb{C}$ and $c \in \mathbb{C}^\nu$ can be chosen arbitrarily.

Proof: It is easily checked that (1.11) represents solutions of (1.10). Conversely, if $x_0(z)$ is any solution of (1.10), then for $c = X^{-1}(z_0)\, x_0(z_0)$ the solution $x(z)$ given by (1.11) satisfies the same initial value condition at z_0 as $x_0(z)$. Their difference satisfies the corresponding homogeneous system, hence is identically zero, owing to Theorem 1 (p. 4). □

The somewhat strange name for (1.11) results from the following observation: For constant $c \in \mathbb{C}^\nu$, the vector $X(z)\, c$ solves the homogeneous system (1.1), so we try an "Ansatz" for the inhomogeneous one by replacing c by a vector-valued function $c(z)$. Differentiation of $X(z)\, c(z)$ and insertion into (1.10) then leads to (1.11).

While (1.11) represents all solutions of (1.10), it requires that we know a fundamental solution of (1.1), and this usually is not the case. In the exercises below, we shall obtain at least local representations, in the form of convergent power series, of solutions of (1.10) without knowing a fundamental solution of (1.1).

Exercises: If nothing else is said, let G be a simply connected region in $\mathbb{C}$ and consider an inhomogeneous system (1.10).

1. Expanding $A(z)$ and $b(z)$ into power series about a point $z_0 \in G$, find the recursion formula for the power series coefficients of solutions.

2. In the case of a constant matrix $A(z) \equiv A$, find a necessary and sufficient condition on A, so that *for every vector polynomial* $b(z)$ a solution of (1.10) exists that is also a polynomial of the same degree as $b(z)$.

3. For
$$B(z) = \begin{bmatrix} 0 & 0 \\ b(z) & A(z) \end{bmatrix},$$
show that $x(z)$ solves (1.10) if and only if $\tilde{x}(z) = [1, x(z)]^T$ solves the homogeneous system $\tilde{x}' = B(z)\, \tilde{x}$ (of dimension $\nu + 1$). Compare this to the next section on reduced systems.

4. For $G = R(0, \rho)$, $\rho > 0$, let $X(z)$ be a fundamental solution of (1.1) with monodromy matrix M. Show that (1.10) has a single-valued

solution $x(z)$, $z \in G$, if and only if we can choose a constant vector c such that for some $z_0 \in G$

$$(\mathrm{e}^{-2\pi i M} - I)c = \int_{z_0}^{z_0 \mathrm{e}^{2\pi i}} X^{-1}(u) b(u)\, du, \tag{1.12}$$

integrating, say, along a circle centered at the origin. Show that a sufficient condition for this to be true is that no nontrivial solution of the homogeneous system (1.1) is single-valued in G.

1.5 Reduced Systems

In this section we shall be concerned with a system (1.1) whose coefficient matrix is triangularly blocked. While corresponding results hold true for upper triangularly blocked matrices, we choose $A(z)$ to have the following *lower triangular* block structure:

$$A(z) = \begin{bmatrix} A_{11}(z) & 0 & \dots & 0 \\ A_{21}(z) & A_{22}(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{\mu 1}(z) & A_{\mu 2}(z) & \dots & A_{\mu\mu}(z) \end{bmatrix}, \tag{1.13}$$

with $\mu \geq 2$, blocks $A_{jk}(z)$ that are holomorphic in a (common) simply connected region G, and such that the diagonal blocks are all square matrices of arbitrary sizes. Such systems will be called *reduced.* If the diagonal blocks of (1.13) are of type $\nu_k \times \nu_k$, we sometimes say that (1.1) *is reduced of type* $(\nu_1, \dots, \nu_\mu)$.

Along with the "large" system (1.1), it is natural to consider the smaller systems

$$x_k' = A_{kk}(z)\, x_k, \qquad 1 \leq k \leq \mu. \tag{1.14}$$

We show that the computation of a fundamental solution of (1.1) is, aside from finitely many integrations, equivalent to computing fundamental solutions for (1.14), for every such k:

Theorem 4 *Given a matrix $A(z)$ as in* (1.13), *the system* (1.1) *has a fundamental solution of the form*

$$X(z) = \begin{bmatrix} X_{11}(z) & 0 & \dots & 0 \\ X_{21}(z) & X_{22}(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ X_{\mu 1}(z) & X_{\mu 2}(z) & \dots & X_{\mu\mu}(z) \end{bmatrix},$$

with $X_{kk}(z)$ being fundamental solutions of (1.14)*, and the off-diagonal blocks $X_{jk}(z)$, for $1 \le k < j \le \mu$, recursively given by*

$$X_{jk}(z) = X_{jj}(z) \left[C_{jk} + \int_{z_0}^{z} X_{jj}^{-1}(u)\, Y_{jk}(u)\, du \right], \quad z \in G, \tag{1.15}$$

with arbitrarily chosen $z_0 \in G$, arbitrary constant matrices C_{jk} of the appropriate sizes, and

$$Y_{jk}(u) = \sum_{\tau=k}^{j-1} A_{j\tau}(u)\, X_{\tau k}(u).$$

Conversely, every lower triangularly blocked fundamental solution of (1.1) *is obtained by* (1.15)*, if the $X_{kk}(z)$ and the constants of integration C_{jk} are appropriately selected.*

Proof: Differentiation of (1.15) and insertion into (1.1) proves that the above $X(z)$ is a fundamental solution of (1.1). If $\tilde{X}(z)$ is any lower triangularly blocked fundamental solution of (1.1), let $X_{kk}(z) = \tilde{X}_{kk}(z)$ and $C_{jk} = X_{jj}^{-1}(z_0)\, \tilde{X}_{jk}(z_0)$, $1 \le k < j \le \mu$. Then $X(z_0) = \tilde{X}(z_0)$, hence $X(z) \equiv \tilde{X}(z)$. □

Observe that in case of $\nu_k = 1$ for every k one can easily compute the diagonal blocks $X_{kk}(z)$ by solving the corresponding scalar ODE. Hence *triangular systems can always be solved.* If one or several ν_k are at least two, this is no longer so, but at least we can say that in a clear sense *reduced systems are easier to solve than general ones.*

Exercises:

1. Verify that the above theorem, with some obvious modifications, generalizes to upper triangularly blocked systems.

2. For $\nu = 2$ and
$$A(z) = \begin{bmatrix} a(z) & 0 \\ b(z) & c(z) \end{bmatrix},$$
with $a(z), b(z), c(z)$ holomorphic in a simply connected region G, explicitly compute a fundamental solution of (1.1), up to finitely many integrations.

3. For $G = R(0, \rho)$, $\rho > 0$, let $A(z)$ as in (1.13) be holomorphic in G, and let $X_{kk}(z)$ be fundamental solutions of (1.14) with monodromy factors C_k, $1 \le k \le \mu$. For $X(z)$ as in the above theorem, show that we can explicitly find a lower triangularly blocked monodromy factor in terms of the integration constants C_{jk} and finitely many definite integrals.

4. Under the assumptions of Exercise 8 on p. 7, show that a computation of a fundamental solution of (1.1) is equivalent to finding a fundamental solution of a system of dimension $\nu - \mu$ and an additional integration.

1.6 Some Additional Notation

It will be convenient for later use to say that a system (1.1) is *elementary* if a matrix-valued holomorphic function $B(z)$, for $z \in G$, exists such that $B'(z) = A(z)$ and $A(z)\,B(z) = B(z)\,A(z)$ hold for every $z \in G$. As was shown in Exercise 3 on p. 7 and the following one, an elementary system has the fundamental solution $X(z) = \exp[B(z)]$; however, such a $B(z)$ does not always exist, so not every system is elementary. Simple examples of elementary systems are those with *constant coefficients*, or systems with diagonal coefficient matrix, or the ones studied in Exercise 2 on p. 7.

In the last century or so, one of the main themes of research has been on the behavior of solutions of (1.1) near an isolated boundary point z_0 of the region G – assuming that such points exist, which implies that G will be multiply connected. In particular, many classical as well as recent results concern the situation where the coefficient matrix $A(z)$ has a pole of order $r+1$ at z_0, and the non-negative integer r then is named the *Poincaré rank* of the system. Relatively little work has been done in case of an essential singularity of $A(z)$ at z_0, and we shall not consider such cases here at all. It is standard to call a system (1.1) a *meromorphic system* on G if the coefficient matrix $A(z)$ is a meromorphic function on G; i.e., every point of G is either a point of holomorphy or a pole of $A(z)$.

In the following chapters we shall study the *local behavior* of solutions of (1.1) near a pole of $A(z)$, and to do so it suffices to take G equal to a punctured disc $R(z_0, \rho)$, for some $\rho > 0$. We shall see that the cases of Poincaré rank $r = 0$, i.e., a simple pole of $A(z)$ at z_0, are essentially different from $r \geq 1$, and we follow the standard terminology in referring to the first resp. second case by saying that (1.1) has a *singularity of first* resp. *of second kind* at z_0. If $A(z)$ is holomorphic in $R(\infty, \rho) = \{z : |z| > \rho\}$, we say, in view of[3] Exercise 2 that (1.1) has rank r at infinity if $B(z) = -z^{-2}A(1/z)$ has rank r at the origin. Accordingly, infinity is a singularity of first, resp. second, kind of (1.1), if $zA(z)$ is holomorphic, resp. has a pole, at infinity. Observe that the same holds when classifying the nature of singularity at the origin! This fact is one of the reasons that, instead of systems (1.1), we shall from now on consider systems obtained by multiplying both sides of (1.1) by z.

[3]Observe that references to exercises within the same section are made by just giving their number.

Exercises: In what follows, let $G = R(z_0, \rho)$, for some $\rho > 0$, and let $A(z)$ be holomorphic in G with at most a pole at z_0. Recall from Section 1.3 that solutions of (1.1) in this case are holomorphic functions on the Riemann surface of $\log(z - z_0)$ over G.

1. For $B(z) = A(z + z_0)$, $z \in R(0, \rho)$, and vector functions $x(z)$, $y(z)$ connected by $y(z) = x(z + z_0)$, show that $x(z)$ is a solution of (1.1) if and only if $y(z)$ solves $y' = B(z)\, y$.

2. For $z_0 = 0$, $B(z) = -z^{-2} A(1/z)$, $z \in R(\infty, 1/\rho) = \{z : |z| > 1/\rho\}$, and $x(z)$, $y(z)$ connected by $y(z) = x(1/z)$, show that $x(z)$ is a solution of (1.1) if and only if $y(z)$ solves $y' = B(z)\, y$.

3. More generally, let G be an arbitrary region, let

$$w = w(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0)$$

be a *Möbius transformation*, and take $\tilde{G}$ as the preimage of G under the (bijective) mapping $z \mapsto w(z)$ of the *compactified complex plane* $\mathbb{C} \cup \{\infty\}$. For simplicity, assume $a/c \notin G$, to ensure $\infty \notin \tilde{G}$. For arbitrary $A(z)$, holomorphic in G, define

$$B(z) = \frac{ad - bc}{(cz + d)^2}\, A(w(z)), \ \ y(z) = x(w(z)), \quad z \in \tilde{G}.$$

Show that $x(z)$ is a solution of (1.1) if and only if $y(z)$ solves $y' = B(z)\, y$.

4. Let $G = R(\infty, \rho)$, let $A(z)$ have Poincaré rank $r \geq 1$ at infinity, and set $ra = \sup_{|z| \geq \rho + \varepsilon} \|z^{1-r} A(z)\|$, for some $\varepsilon > 0$. For

$$\bar{S} = \{z : \ |z| \geq \rho + \varepsilon, \ \alpha \leq \arg z \leq \beta\},$$

with arbitrary $\alpha < \beta$, show that for every fundamental solution $X(z)$ of (1.1) one can find $c > 0$ so that

$$\|X(z)\| \leq c\, \mathrm{e}^{a|z|^r}, \quad z \in \bar{S}.$$

5. For every dimension $\nu \geq 1$, find an elementary system of Poincaré rank $r \geq 1$ at infinity for which the estimate in the previous exercise is sharp.

6. For every dimension $\nu \geq 2$, find an elementary system of Poincaré rank $r \geq 1$ at infinity for which fundamental solutions only grow like a power of z; hence the estimate in Exercise 4 *is not sharp.* Check that for $\nu = 1$ the estimate always is sharp.

7. Consider a system (1.1) that is meromorphic in G, and let z_0 be a pole of $A(z)$. If it so happens that a fundamental solution exists which only has a removable singularity at z_0, we say that z_0 is *an apparent singularity of* (1.1). Check that then *every fundamental solution* $X(z)$ must be holomorphic at z_0, but $\det X(z_0) = 0$.

8. For every dimension $\nu \geq 1$, find a system (1.1) that is meromorphic in some region G, with infinitely many apparent singularities in G.

2

Singularities of First Kind

Throughout this chapter, we shall be concerned with a system (1.1) (p. 2) having a singularity of first kind, i.e., a pole of first order, at some point z_0, and we wish to study the behavior of solutions near this point. In particular, we wish to solve the following problems as explicitly as we possibly can:

P1) Given a fundamental solution $X(z)$ of (1.1), find a monodromy matrix at z_0; i.e., find M so that $X(z) = S(z)\,(z-z_0)^M$, with $S(z)$ holomorphic and single-valued in $0 < |z - z_0| < \rho$, for some $\rho > 0$.

P2) Determine the kind of singularity that $S(z)$ has at z_0; i.e., decide whether this singularity is *removable, or a pole, or an essential one.*

P3) Find the coefficients in the Laurent expansion of $S(z)$ about the point z_0, or more precisely, find equations that allow the computation of at least finitely many such coefficients.

The following observations are very useful in order to simplify the investigations we have in mind:

- Suppose that $X(z) = S(z)\,(z-z_0)^M$ is *some* fundamental solution of (1.1), then according to Exercise 1 on p. 7 every other fundamental solution is obtained as $\tilde{X}(z) = X(z)\,C = \tilde{S}(z)\,(z-z_0)^{\tilde{M}}$, with $\tilde{S}(z) = S(z)\,C$, $\tilde{M} = C^{-1}M\,C$, for a unique invertible matrix C. Therefore we conclude that it suffices to solve the above problems for *one particular fundamental solution* $X(z)$.

- We shall see that *for singularities of first kind the matrix $S(z)$ never has an essential singularity at z_0*. Whether it has a pole or a removable one depends on the selection of the monodromy matrix, since instead of M we can also choose $M - kI$, for every integer k, and accordingly replace $S(z)$ by $z^k S(z)$. Hence in a way, poles and removable singularities of $S(z)$ should not really be distinguished in this context. It will, however, make a difference *whether or not $S(z)$ has a removable singularity and at the same time* $\det S(z)$ *does not vanish at z_0*, meaning that the power series expansion of $S(z)$ begins with an invertible constant term.

- According to the exercises in Section 1.6, we may without loss in generality make z_0 equal to any preassigned point in the compactified complex plane $\mathbb{C} \cup \{\infty\}$. For singularities of first kind it is customary to choose $z_0 = 0$, and we shall follow this convention. Moreover, we also adopt the custom to consider the differential operator $z\,(d/dz)$ instead of just the derivative d/dz; this has advantages, e.g., when making a change of variable $z = 1/u$ (see Section 1.6). As a consequence, we shall here consider a system of the form

$$z\,x' = A(z)\,x, \quad A(z) = \sum_{n=0}^{\infty} A_n\, z^n, \quad |z| < \rho. \tag{2.1}$$

  Hence in other words, $A(z)$ is a holomorphic matrix function in $D(0,\rho)$, with $\rho > 0$, and we shall have in mind that $A_0 \neq 0$, although all results remain correct for $A_0 = 0$ as well.

As we shall see, the following condition upon the spectrum of A_0 will be very important:

E) We say that (2.1) has *good spectrum* if no two eigenvalues of A_0 differ by a natural number, or in other words, if $A_0 + nI$ and A_0, for every $n \in \mathbb{N}$, have disjoint spectra. *Observe that we do not regard* 0 *as a natural number; thus it may be that A_0 has equal eigenvalues!*

For systems with good spectrum we shall see that A_0 will be a monodromy matrix for some fundamental solution $X(z)$, and we shall obtain a representation for $X(z)$ from which we can read off its behavior at the origin. For the other cases we shall obtain a similar, but more complicated result.

The theory of singularities of first kind is covered in most books dealing with ODE in the complex plane. In addition to those mentioned in Chapter 1, one can also consult *Deligne* [84] and *Yoshida* [290]. In this chapter, we shall also introduce some of the *special functions* which have been studied in the past. For more details, and other functions which are not mentioned here, see the books by *Erdélyi* [100], *Schäfke* [235], *Magnus, Oberhettinger, and Soni* [180], and *Iwasaki, Kimura, Shimomura, and Yoshida* [141].

2.1 Systems with Good Spectrum

Here we prove a well-known theorem saying that for systems with good spectrum the matrix A_0 always is a monodromy matrix. Moreover, a fundamental solution can in principle be computed in a form from which the behavior of solutions near the origin may be deduced:

Theorem 5 *Every system* (2.1) *with good spectrum has a unique fundamental solution of the form*

$$X(z) = S(z)\, z^{A_0}, \quad S(z) = \sum_{n=0}^{\infty} S_n\, z^n, \; S_0 = I, \qquad |z| < \rho. \tag{2.2}$$

The coefficients S_n *are uniquely determined by the relations*

$$S_n\,(A_0 + nI) - A_0\, S_n = \sum_{m=0}^{n-1} A_{n-m}\, S_m, \qquad n \ge 1. \tag{2.3}$$

Proof: Inserting the "Ansatz" (2.2) into (2.1) and comparing coefficients easily leads to the recursion equations (2.3). Lemma 24 (p. 212) implies that the coefficients S_n are uniquely determined by (2.3), owing to assumption **E**. Hence we are left to show that the resulting power series for $S(z)$ converges as desired. To do this, we proceed similarly to the proof of Lemma 1 (p. 2): We have $\|A_n\| \le c\,K^n$ for every constant $K > 1/\rho$ and sufficiently large $c > 0$, depending upon K. Abbreviating the right-hand side of (2.3) by B_n, we obtain $\|B_n\| \le c \sum_{m=0}^{n-1} K^{n-m}\, \|S_m\|,\ n \ge 1$. Divide (2.3) by n and think of the elements of S_n arranged, in one way or another, into a vector of length ν^2. Doing so, we obtain a linear system of equations with a coefficient matrix of size $\nu^2 \times \nu^2$, whose entries are bounded functions of n. Its determinant tends to 1 as $n \to \infty$, and is never going to vanish, according to **E**. Consequently, the inverse of the coefficient matrix also is a bounded function of n. These observations imply the estimate $\|S_n\| \le n^{-1}\tilde{c}\|B_n\|,\ n \ge 1$, with sufficiently large $\tilde{c}$, independent of n. Let $s_0 = \|S_0\|,\ s_n = n^{-1}\hat{c} \sum_{m=0}^{n-1} K^{n-m}\, s_m,\ n \ge 1$, with $\hat{c} = \tilde{c}\,c$, and conclude by induction $\|S_n\| \le s_n,\ n \ge 0$. The power series $f(z) = \sum_0^{\infty} s_n\, z^n$ formally satisfies $f'(z) = \hat{c}K\, f(z)\,(1 - Kz)^{-1}$, and as in the proof of Lemma 1 (p. 2) we obtain convergence of $f(z)$ for $|z| < K^{-1}$, hence convergence of $S(z)$ for $|z| < \rho$. □

Note that the above theorem coincides with Lemma 1 (p. 2) in case of $A_0 = 0$, which trivially has good spectrum. Moreover, the theorem obviously solves the three problems stated above in quite a satisfactory manner: The computation of a monodromy matrix is trivial, the corresponding $S(z)$ has a removable singularity at the origin, $\det S(z)$ attains value 1 there, and the coefficients of its Laurent expansion, which here is a power series, can be recursively computed from (2.3). As we shall see, the situation gets more complicated for systems with general spectrum: First of all, A_0 will no longer be a monodromy matrix, although closely related to one, and secondly the single-valued part of fundamental solutions has a somewhat more complicated structure as well. Nonetheless, we shall also be able to completely analyze the structure of fundamental solutions in the general situation.

Exercises: In the following exercises, consider a fixed system (2.1) with good spectrum.

1. Give a different proof for the existence part of Theorem 5 as follows: For $N \in \mathbb{N}$, assume that we computed $S_1, \ldots, S_N$ from (2.3), and let $P_N(z) = I + \sum_1^N S_n\, z^n$, $B(z) = A(z)\, P_N(z) - zP_N'(z) - P_N(z)\, A_0$, $\tilde{X}(z) = X(z) - P_N(z)\, z^{A_0}$. For sufficiently large N, show that $X(z)$ solves (2.1) if and only if
$$\tilde{X}(z) = \int_0^z \Big[B(u)\, u^{A_0} + A(u)\, \tilde{X}(u) \Big] \frac{du}{u}. \tag{2.4}$$
Then, by the standard iteration method, show that (2.4) has a solution $\tilde{X}(z) = \tilde{S}(z)\, z^{A_0}$, with $\tilde{S}(z)$ holomorphic near the origin and vanishing of order at least $N+1$.

2. Let T be invertible, so that $J = T^{-1} A_0\, T$ is in Jordan canonical form. Show that (2.1) has a fundamental solution $X(z) = S(z)\, z^J$, with $S(z) = T + \sum_{n=1}^\infty S_n\, z^n$, $|z| < \rho$.

3. Let s_0 be an eigenvector of A_0, corresponding to the eigenvalue μ. Show that (2.1) has a solution $x(z) = s(z)\, z^\mu$, with $s(z) = s_0 + \sum_{n=1}^\infty s_n\, z^n$, $|z| < \rho$. Such solutions are called *Floquet solutions*, and we refer to μ as the corresponding *Floquet exponent*. Find the recursion formulas for the coefficients s_n.

4. Show that (2.1) has k linearly independent Floquet solutions if and only if A_0 has k linearly independent eigenvectors. In particular, (2.1) has a fundamental solution consisting of Floquet solutions if and only if A_0 is diagonalizable.

5. In dimension $\nu = 2$, let
$$A_0 = \begin{bmatrix} \mu & 0 \\ 1 & \mu \end{bmatrix}, \qquad \mu \in \mathbb{C}.$$
Show that (2.1) has a fundamental solution consisting of one Floquet solution and another one of the form $x(z) = (s_1(z) + s_2(z)\log z)\, z^\mu$, $s_j(z) = \sum_0^\infty s_n\, z^n$, $|z| < \rho$. Try to generalize this to higher dimensions.

2.2 Confluent Hypergeometric Systems

As an application of the results of Section 2.1, we study in more detail the very special case of

$$zx' = (zA + B)\,x, \quad A, B \in \mathbb{C}^{\nu\times\nu}. \tag{2.5}$$

We shall refer to this case as the *confluent hypergeometric system*, since it may be considered as a generalization of the second-order scalar ODE bearing the same name, introduced in Exercise 3. Under various additional assumptions on A and B, such systems, and/or the closely related *hypergeometric systems* that we shall look at in the next section, have been studied, e.g., by *Jurkat, Lutz, and Peyerimhoff* [147, 148], *Okubo and Takano* [207], *Balser, Jurkat, and Lutz* [37, 41], *Kohno and Yokoyama* [161], *Balser* [11–13, 20], *Schäfke* [240], *Okubo, Takano, and Yoshida* [208], and *Yokoyama* [288, 289].

For simplicity we shall here restrict our discussion to the case where B is diagonalizable and **E** holds. So according to Exercise 4 on p. 20 we have ν linearly independent Floquet solutions $x(z) = \sum_{n=0}^{\infty} s_n\, z^{n+\mu}$, where μ is an eigenvalue of B and s_0 a corresponding eigenvector, and the series converges for every $z \in \mathbb{C}$. The coefficients satisfy the following simple recursion relation:

$$s_n = ((n+\mu)I - B)^{-1}\, A\, s_{n-1}, \quad n \ge 1. \tag{2.6}$$

Note that the inverse matrix always exists according to **E**. Hence we see that s_n is a product of finitely many matrices times s_0. To further simplify (2.6), we may even assume that B is, indeed, a diagonal matrix $D = \operatorname{diag}[\mu_1, \dots, \mu_\nu]$, since otherwise we have $B = TDT^{-1}$ for some invertible T, and setting $A = T\tilde{A}T^{-1}$, $s_n = T\tilde{s}_n$, this leads to a similar recursion for $\tilde{s}_n$. Then, μ is one of the values μ_k and s_0 a corresponding unit vector.

Despite of the relatively simple form of (2.6), we will have to make some severe restrictions before we succeed in computing s_n in closed form. Essentially, there are two cases that we shall now present.

To begin, consider (2.6) in the special case of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \tag{2.7}$$

assuming that $\mu_1 - \mu_2 \ne \mathbb{Z}$ except for $\mu_1 = \mu_2$, so that **E** holds. In this case, let us try to explicitly compute the Floquet solution corresponding to the exponent $\mu = \mu_1$; the computation of the other one follows the same lines. Denoting the two coordinates of s_n by f_n, g_n, we find that (2.6) is equivalent to

$$nf_n = af_{n-1} + bg_{n-1}, \;\; (n+\beta)g_n = cf_{n-1} + dg_{n-1}, \quad n \ge 1,$$

for $\beta = \mu_1 - \mu_2$, and the initial conditions $f_0 = 1,\ g_0 = 0$. Note $n + \beta \neq 0,\ n \geq 1$, according to **E**. This implies

$$\begin{aligned}(n+1)(n+\beta) f_{n+1} &= (n+\beta)(a f_n + b g_n) \\ &= a(n+\beta) f_n + b(c f_{n-1} + d g_{n-1}).\end{aligned}$$

Using the original relations, we can eliminate g_{n-1} to obtain the following second order recursion for the sequence (f_n):

$$(n+1)(n+\beta) f_{n+1} = [n(a+d) + a\beta] f_n - (ad - bc) f_{n-1}, \quad n \geq 1,$$

together with the initial conditions $f_0 = 1,\ f_1 = a$. Unfortunately, such a recursion in general is still very difficult to solve – however, if $ad - bc = \det A$ would vanish, this would reduce to a first-order relation. Luckily, there is a little trick to achieve this: Substitute[1] $x = e^{\lambda z} \tilde{x}$ into the system (2.1) to obtain the equivalent system $z\tilde{x}' = (A(z) - z\lambda)\,\tilde{x}$. In case of a confluent hypergeometric system and λ equal to an eigenvalue of A, we arrive at another such system with $\det A = 0$. Note that if we computed a Floquet solution of the new system, we then can reverse the transformation to obtain such a solution for the original one.

To proceed, let us now assume $ad - bc = 0$; hence one eigenvalue of A vanishes. Then $\lambda = a + d$ is equal to the second, possibly nonzero, eigenvalue, and the above recursion becomes

$$(n+1)(n+\beta) f_{n+1} = (n\lambda + a\beta) f_n, \quad n \geq 1,$$

with $f_1 = a$. Leaving the case of $\lambda = 0$ as an exercise and writing $\alpha = a\beta/\lambda$ for $\lambda \neq 0$, we find[2]

$$\left.\begin{aligned} f_n &= \frac{\lambda^n\,(\alpha)_n}{n!\,(\beta)_n} \\ g_n &= c\frac{\lambda^{n-1}(\alpha+1)_{n-1}}{(n-1)!\,(\beta+1)_n} \end{aligned}\right\}, \quad n \geq 1.$$

Thus, one can explicitly express the Floquet solution of the confluent hypergeometric system in terms of the following well-known higher transcendental function:

CONFLUENT HYPERGEOMETRIC FUNCTION

For $\alpha \in \mathbb{C},\ \beta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, the function

$$F(\alpha; \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!\,(\beta)_n} z^n, \quad z \in \mathbb{C},$$

---

[1]Note that what is done here is a trivial case of what will be introduced as analytic transformations in the following section.

[2]Here we use the *Pochhammer symbol* $(\alpha)_0 = 1,\ (\alpha)_n = \alpha \cdot \ldots \cdot (\alpha + n - 1),\ n \geq 1$.

is called *confluent hypergeometric function.* Another name for this function is *Kummer's function.* It arises in solutions of the *confluent hypergeometric differential equation* introduced in the exercises below. For $\alpha = -m$, $m \in \mathbb{N}_0$, the function is a polynomial of degree m; otherwise, it is an entire function of exponential order 1 and finite type.

In the case of $\lambda = 0$, the coefficients f_n obviously decrease at a much faster rate. This is why the corresponding functions are of smaller exponential order. In a way, it is typical in the theory of linear systems of meromorphic ODE to have a "generic situation" (here: $\lambda \neq 0$ and $\alpha \neq 0, -1, \ldots$) in which solutions show a certain behavior (here, they are entire functions of exponential order 1 and finite type), while in the remaining case they are essentially different (of smaller order, or even polynomials). To explicitly find the solutions in these exceptional cases, we define another type of special functions, which are very important in applications:

Bessel's Function

For $\mu \in \mathbb{C}$, the function

$$J_\mu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\mu+n)} (z/2)^{2n+\mu}, \quad z \in \mathbb{C},$$

is called *Bessel's function.* Removing the power $(z/2)^\mu$, we obtain an entire function of exponential order 1 and finite type. The function is a solution of a scalar second-order ODE, called *Bessel's differential equation.*

In Exercise 2 we shall show that Bessel's function also arises in solutions of (2.6) for $\nu = 2$ and nilpotent A, i.e., $\lambda = 0$.

Next, we briefly mention another special case of (2.5) where Floquet solutions can be computed in closed form: For arbitrary dimension ν, let

$$A = \operatorname{diag}[0, \ldots, 0, 1], \quad B = \begin{bmatrix} \lambda'_1 & 0 & \ldots & 0 & a_1 \\ 0 & \lambda'_2 & \ldots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda'_{\nu-1} & a_{\nu-1} \\ b_1 & b_2 & \ldots & b_{\nu-1} & \lambda'_\nu \end{bmatrix}. \tag{2.8}$$

Under certain generic additional assumptions one can explicitly compute Floquet solutions of (2.5), using the following well-known functions:

Generalized Confluent Hypergeometric Functions

For $m \geq 1$, $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $1 \leq j \leq m$, the function

$$F(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_m)_n}{n!\,(\beta_1)_n \cdot \ldots \cdot (\beta_m)_n} z^n$$

(with radius of convergence of the series equal to infinity), is called *generalized confluent hypergeometric function*. For some parameter values (which?), the function is a polynomial, and otherwise an entire function of exponential order 1 and finite type. The function arises in solutions of a scalar $(m+1)$st-order ODE, called *generalized confluent hypergeometric differential equation*, introduced in the exercises below.

For more details upon this case we refer to [13].

Exercises: In the first two of the following exercises, consider a fixed confluent hypergeometric system with A, B as in (2.7). Also, restrict $\mu_1-\mu_2$ so that **E** holds. For the last one, consider $m \in \mathbb{N}$ and parameters α_j, β_j as in the definition of the generalized confluent hypergeometric function, and let δ denote the differential operator $z(d/dz)$.

1. For A having distinct eigenvalues, but *not* assuming $\det A = 0$, explicitly express one Floquet solution of (2.5) using exponential and confluent hypergeometric functions.

2. For A having equal eigenvalues, do the same as above in terms of exponential and Bessel functions.

3. Verify that the (scalar) *confluent hypergeometric* ODE
$$z\,y'' - (z-\beta)y' - \alpha y = 0$$
for $\beta \neq 0, -1, -2, \ldots$ has the solution $F(\alpha;\beta;z)$. Moreover, verify that the equation has *only one solution, aside from a constant factor,* which is holomorphic at the origin. Note that in the literature one can find another second-order ODE, closely related to the one above, bearing the same name.

4. Show *Kummer's transformation*
$$F(\alpha;\beta;z) = \mathrm{e}^z\, F(\beta-\alpha;\beta;-z), \quad z \in \mathbb{C},\ \beta \neq 0, -1, -2, \ldots$$

5. For $\operatorname{Re}\beta > \operatorname{Re}\alpha > 0$, show
$$F(\alpha;\beta;z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)} \int_0^1 \mathrm{e}^{zt} t^{\alpha-1}(1-t)^{\beta-\alpha-1}\,dt.$$

6. Verify that *Bessel's equation*
$$z^2 y'' + z y' + (z^2 - \mu^2) y = 0$$
has the solution $J_\mu(z)$.

7. Show that $F(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_m; z)$ satisfies the following scalar ODE of $(m+1)$st order, called the *confluent hypergeometric differential equation*:

$$[(\delta + \alpha_1) \cdot \ldots \cdot (\delta + \alpha_m) - (\delta + \beta_1) \cdot \ldots \cdot (\delta + \beta_m) \frac{d}{dz}]\, y = 0.$$

2.3 Hypergeometric Systems

In this section we consider so-called *hypergeometric systems*, which are of the form

$$(A - zI)\, x' = B\, x, \quad A, B \in \mathbb{C}^{\nu \times \nu}. \tag{2.9}$$

As shall become clear in the following chapters, (2.9) is intimately related, via Laplace transform, to a confluent hypergeometric system with slightly different A and B. Another relation is through a process called *confluence*, which shall not be discussed here but is mentioned to explain the name for the systems (2.5).

For simplicity, we restrict our discussion to $A = \operatorname{diag}[\lambda_1, \ldots, \lambda_\nu]$, with not necessarily distinct values λ_k. The system (2.9) then is meromorphic in $\mathbb{C}$ with first-order poles at the numbers $\lambda_1, \ldots, \lambda_\nu$. If all the λ_k are equal, a change of variable $z = u+\lambda$, reduces the system to one treated in Exercise 1 on p. 9, so we exclude this case here. For $\lambda_1 = \ldots = \lambda_{\nu-1} \neq \lambda_\nu$, we shall compute Floquet solutions of (2.9) using the *hypergeometric function* resp. its generalized versions. For $\nu = 3$ and three distinct values λ_k, the system, under some additional assumptions, should be closely related to *Heun's ODE*, and the corresponding confluent one to one of its various confluent forms [234], but to the author's knowledge these relations have never been worked out. We leave this to the reader as a research problem.

Making a change of variable $z = au + b$, $a \neq 0$, one can achieve $\lambda_1 = 0$, $\lambda_\nu = 1$. This shows that in dimension $\nu = 2$ we may assume

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Putting the system into the form (2.1), we find that the Floquet exponents are $-a$ and 0. For simplicity we restrict to a noninteger value for a, so that in particular **E** is satisfied. Using the same notation as in the previous section, the recursion relations for the coefficients of a Floquet solution with exponent $\mu = -a$ are

$$0 = n f_n + b\, g_n, \;\; (n+1-a) g_{n+1} = c\, f_n + (n+d-a) g_n, \;\; n \geq 0,$$

with $f_0 = 1$, $g_0 = 0$. Eliminating f_n, we obtain $n(n+1-a)\,g_{n+1} = (n+\alpha)(n+\beta)\,g_n$, $n \geq 1$, with $\alpha+\beta = d-a$, $\alpha\beta = -cb$ and the initial condition $g_1 = c/(1-a)$. With $\gamma = 1-a$, this implies

$$\left.\begin{array}{rcl} f_n & = & \frac{(\alpha)_n\,(\beta)_n}{n!\,(\gamma)_n} \\[2ex] g_n & = & c\frac{(1+\alpha)_{n-1}\,(1+\beta)_{n-1}}{(n-1)!\,(\gamma)_n} \end{array}\right\} \quad n \geq 1.$$

So we can compute the corresponding Floquet solution using the perhaps most famous function we define in this chapter:

Hypergeometric Function

For $\alpha, \beta \in \mathbb{C}$, $\gamma \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, the function

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{n!\,(\gamma)_n} z^n, \quad |z| < 1,$$

is called *hypergeometric function.* It is a solution of a scalar second-order ODE, called *hypergeometric differential equation*, introduced in the exercises below.

For arbitrary dimension ν, a special case where one can compute Floquet solutions in closed form occurs for A, B as in (2.8), under some additional generic assumptions upon the parameters. For details we again refer to [13]; here we only mention that for these calculations one has to use the following other well-known special functions:

Generalized Hypergeometric Functions

For $m \geq 1$, $\alpha_j \in \mathbb{C}$, $1 \leq j \leq m$, and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $1 \leq j \leq m-1$, the function

$$F(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_{m-1}; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_m)_n}{n!\,(\beta_1)_n \cdot \ldots \cdot (\beta_{m-1})_n} z^n$$

(with radius of convergence of the series at least equal to 1), is called *generalized hypergeometric function.* It arises in solutions of a scalar mth-order ODE, called *generalized hypergeometric differential equation*, introduced in the exercises below.

Exercises: In the following exercises, let α, β, γ, resp. α_j, β_j be as in the definition of the hypergeometric, resp. generalized hypergeometric functions, and let δ denote the differential operator $z(d/dz)$.

1. Verify that the *hypergeometric* ODE

$$z(1-z)\,y'' + (\gamma - (\alpha+\beta+1)z)\,y' - \alpha\beta\,y = 0$$

has the solution $F(\alpha, \beta; \gamma; z)$, $|z| < 1$.

2. Show that the function $F(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_{m-1}; z)$ satisfies the following scalar ODE of mth order, called the *generalized hypergeometric differential equation*:

$$[(\delta + \alpha_1) \cdot \ldots \cdot (\delta + \alpha_m) - (\delta + \beta_1) \cdot \ldots \cdot (\delta + \beta_{m-1}) \frac{d}{dz}]\, y = 0.$$

3. Use Theorem 1 (p. 4) to conclude that $F(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_{m-1}; z)$ can be holomorphically continued into the complex plane with a single cut from 1 to infinity, which usually is made along the positive real axis.

4. Show

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\, \Gamma(\gamma - \beta)} \int_0^1 \frac{(1-t)^{\gamma-\beta-1} t^{\beta-1}}{(1-zt)^{\alpha}}\, dt, \qquad (2.10)$$

for $z \in \mathbb{C} \setminus [1, \infty)$ and $\mathrm{Re}\, \gamma > \mathrm{Re}\, \beta > 0$.

5. For $\mathrm{Re}\,(\gamma - \alpha - \beta) > 0$, show

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n\, (\beta)_n}{n!\, (\gamma)_n} = \frac{\Gamma(\gamma)\, \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\, \Gamma(\gamma - \beta)}. \qquad (2.11)$$

6. Show $F(\alpha, \beta; \gamma; z) = (1 - z)^{\gamma - \alpha - \beta}\, F(\gamma - \alpha, \gamma - \beta; \gamma; z)$.

2.4 Systems with General Spectrum

Throughout this section, let a system (2.1) with general spectrum be given. We shall solve the same three problems stated at the beginning of this chapter; however, the solutions will be less direct, and some modifications in the statements and proofs of the results are necessary. Our treatment is very similar to *Gantmacher*'s [105], although the proofs are of a slightly different flavor.

The main tools in this section will be so-called analytic transformations, introduced and studied by *Birkhoff* [54]:

A square matrix-valued function $T(z)$ will be called an *analytic transformation* if it is holomorphic in a neighborhood of the origin, and so that $\det T(z) \neq 0$ there. This obviously is equivalent to $T(z) = \sum_0^\infty T_n\, z^n$ for $|z|$ sufficiently small, and $\det T_0 \neq 0$. If $T_n = 0$ for every $n \geq 1$, we shall sometimes speak of a *constant transformation*.

Given an analytic transformation $T(z)$, we set $x = T(z)\, \tilde{x}$. Then x is a solution of (2.1) if and only if $\tilde{x}$ solves

$$z \tilde{x}' = B(z)\, \tilde{x}, \qquad (2.12)$$

with $B(z)$ given by

$$zT'(z) = A(z)\,T(z) - T(z)\,B(z). \tag{2.13}$$

Given $T(z)$ and (2.1), we refer to (2.12) as the *transformed system*. The matrix $B(z)$ is again holomorphic near the origin, but possibly on a smaller circle. In particular, the transformed system has a singularity of first kind there. For later use, we mention that in case of $A(z)$ having a pole of order r at the origin, then so does $B(z)$. Therefore, analytic transformations *preserve the Poincaré rank* of systems at the origin. Note that we shall use the term "analytic transformation" both for the matrix $T(z)$ and for the change of variable $x = T(z)\,\tilde{x}$, but we think this will not lead to confusion.

The special form of $T(z)$ implies that solutions of both systems (2.1) and (2.12) behave the same near the origin, e.g., show the same monodromy behavior. This is why we call two systems (2.1) and (2.12) *analytically equivalent*, if we can find an analytic transformation $T(z)$ satisfying (2.13). Note that (2.13), for given $A(z)$ and $B(z)$, is nothing but a linear system of ODE of Poincaré rank $r = 1$ for the entries of $T(z)$. To check analytic equivalence requires finding a solution $T(z)$ that is holomorphic at the origin, with $T(0)$ invertible.

Given (2.1), we shall see that we can construct an analytic transformation so that the task of finding solutions of the transformed system is easier than for the original one. We have already seen this happen on p. 21, where a transformation $x = \mathrm{e}^{\lambda z}\tilde{x}$ helped to find solutions of the two-dimensional confluent hypergeometric system. Theorem 5 (p. 19) may be restated as saying that systems with good spectrum are analytically equivalent to (2.12) with $B(z) \equiv A_0$, which is an elementary system with fundamental solution z^{A_0}. For general spectrum we shall prove that (2.1) is analytically equivalent to a reduced system in the sense of Section 1.5, with diagonal blocks which all have good spectrum. To do so, we show the following lemma, concerning a nonlinear system of ODE of a very special form:

Lemma 2 *Given a system* (2.1), *assume that $A(z)$ can be blocked as*

$$A(z) = \begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix},$$

so that $A_{12}(0) = 0$, and that the square matrices $A_{11}(0)$ and $A_{22}(0)+nI$, for every $n \in \mathbb{N}$, have disjoint spectra. Then for sufficiently small $\tilde{\rho} > 0$ there exists a unique matrix-valued holomorphic function $T_{12}(z)$, with $T_{12}(0) = 0$ and

$$\begin{aligned} zT_{12}'(z) &= A_{11}(z)\,T_{12}(z) - T_{12}(z)\,A_{22}(z) \\ &\quad -T_{12}(z)\,A_{21}(z)\,T_{12}(z) + A_{12}(z), \quad |z| < \tilde{\rho}. \end{aligned} \tag{2.14}$$

Proof: Expanding $A_{jk}(z) = \sum_0^\infty A_n^{(jk)}\, z^n$, $T_{12}(z) = \sum_1^\infty T_n\, z^n$ and inserting into (2.14), we obtain the recursions

$$\begin{aligned} T_n(A_0^{(22)} + nI) - A_0^{(11)}\, T_n &= \sum_{m=1}^{n-1} (A_{n-m}^{(11)}\, T_m - T_m\, A_{n-m}^{(22)}) \\ &\quad + A_n^{(12)} - \sum_{\mu=1}^{n-1} T_\mu \sum_{m=1}^{n-\mu} A_{n-\mu-m}^{(21)}\, T_m, \end{aligned}$$

for $n \ge 1$. According to our assumption and Lemma 24 (p. 212), these recursions determine the T_n uniquely, so to prove existence we are left to show convergence of the power series so obtained for $T(z)$. To do this we proceed analogously to the proof of Theorem 5 (p. 19): Defining [3]

$$t_n = c\, K^n \Big\{ 1 + \sum_{\mu=1}^{n-1} K^{-\mu} t_\mu \Big[2 + \sum_{m=1}^{n-\mu} K^{-m} t_m \Big] \Big\}, \quad n \ge 1,$$

for $K > 1/\rho$ and sufficiently large $c > 0$, we show by induction $\|T_n\| \le t_n$, $n \ge 1$. Putting $f(z) = \sum_1^\infty t_n\, (z/K)^n$, one can formally obtain the quadratic equation $(1-z)\, f(z) = zc\,[1 + 2f(z)] + c\, f^2(z)$. Its solutions are

$$f_\pm(z) = (2c)^{-1} \Big\{ 1 - z(2c+1) \pm \sqrt{[1 - z(2c+1)]^2 - 4c^2 z}\, \Big\},$$

which both are holomorphic near the origin. Moreover, $f_-(0) = 0$, and the coefficients of its power series expansion satisfy the same recursion as, so are in fact equal to, the numbers $t_n K^{-n}$. This proves convergence of $\sum_1^\infty t_n\, z^n$, for sufficiently small $|z|$. □

Note that, unlike for linear systems, we have not shown that the matrix $T(z)$ in the above lemma is holomorphic in the same disc as the coefficient functions; in general, this is not true for nonlinear systems, as one can learn from simple examples.

Lemma 2 may be rephrased as showing existence of a unique analytic transformation of the form

$$T(z) = \begin{bmatrix} I & T_{12}(z) \\ 0 & I \end{bmatrix}, \qquad T_{12}(0) = 0,$$

transforming (2.1) into (2.12), with

$$B(z) = \begin{bmatrix} B_{11}(z) & 0 \\ B_{21}(z) & B_{22}(z) \end{bmatrix},$$

[3] Observe that the same recursion, but with c replaced by c/n, would also ensure $\|T_n\| \le t_n$, and this majorizing sequence would clearly give a better estimate. However, then the corresponding function $f(z)$ is a solution of a Riccati differential equation, instead of a simple quadratic equation. Hence proving analyticity is more complicated.

$$B_{11}(z) = A_{11}(z) - T_{12}(z)\,A_{21}(z), \quad B_{21}(z) = A_{21}(z),$$
$$B_{22}(z) = A_{21}(z)\,T_{12}(z) + A_{22}(z).$$

Iterating this result, we are able to prove that a system with general spectrum can be transformed into another one that is reduced in a block structure determined by the Jordan canonical form of A_0:

Proposition 3 *Let a system* (2.1) *with general spectrum be given. Assume that the matrix A_0 is diagonally blocked, with the diagonal blocks having exactly one eigenvalue, and let these eigenvalues have increasing real parts. Then an analytic transformation $T(z)$ exists that is upper triangularly blocked with respect to the block structure of A_0, with diagonal blocks identically equal to I, so that the corresponding transformed system is lower triangularly reduced in the same block structure.*

Proof: We proceed by induction with respect to the number of blocks of A_0. In the case of one such block nothing remains to prove, while for two blocks Lemma 2 assures the existence of the transformation as stated. If we have even more diagonal blocks in A_0, we block all matrices in a coarser block structure with two diagonal blocks by grouping several of the blocks of A_0 into one large block. Then we can again apply Lemma 2 to obtain a system that is lower triangularly blocked of this coarser type, with two diagonal blocks to which the induction hypothesis applies. Since diagonally blocked analytic transformations leave the system in lower triangularly blocked form, we can use such a transformation to put all the diagonal blocks into the desired form. This, however, completes the proof. □

Note that the assumptions of the above proposition can always be made to hold by a constant transformation $T(z) \equiv T$, putting A_0 into Jordan canonical form with the eigenvalues ordered accordingly. Hence, in principle the above proposition allows us to compute fundamental solutions of systems with general spectrum: First, we find an analytic transformation to another system that is reduced and has diagonal blocks, each of which has good spectrum. Then, we compute a fundamental solution of the new, reduced, system by means of Theorem 5, applied to each diagonal block, and Theorem 4. This fundamental solution will be investigated further:

Proposition 4 *Let a system* (2.1) *be given that is reduced of some type $(\nu_1, \ldots, \nu_\mu)$, with $A_{jj}(0)$ having exactly one eigenvalue λ_j, and assume that these eigenvalues have increasing real parts. Moreover, let $A_{kj}(0) = 0$ for $k > j$. Finally, let $\kappa_j \in \mathbb{Z}$ be such that $0 \le \operatorname{Re}\lambda_j - \kappa_j < 1$. Then there exists a triangularly blocked fundamental solution $X(z)$ of* (2.1) *of the form*

$$X(z) = T(z)\,z^K\,z^M,$$

where $T(z)$ is a lower triangularly blocked analytic transformation, M is a lower triangularly blocked constant matrix with diagonal blocks equal to $A_{jj}(0) - \kappa_j I$, and $K = \operatorname{diag}[\kappa_1 I_{\nu_1}, \ldots, \kappa_\mu I_{\nu_\mu}]$.

Proof: Using Theorem 5 (p. 19), we compute fundamental solutions $X_{jj}(z) = T_{jj}(z)\, z^{A_{jj}(0)}$ for the diagonal blocks of our system, and then compute the blocks below the diagonal using Theorem 4 (p. 12), thus obtaining some lower triangularly blocked fundamental solution $X(z)$. Obviously, $X(z)$ has a monodromy factor that is blocked in the same way, and according to Theorem 71 (p. 241), so is the unique monodromy matrix M with eigenvalues in $[0, 1)$. Its diagonal blocks M_{jj} are monodromy matrices for the diagonal blocks of $X(z)$, having eigenvalues with real parts in $[0, 1)$. Another such monodromy matrix is $A_{jj}(0) - \kappa_j I$; hence, according to the uniqueness part of Theorem 71, we have $M_{jj} = A_{jj}(0) - \kappa_j I$. Defining $T(z) = X(z)\, z^{-M}\, z^{-K}$, we conclude that $T(z)$ is single-valued at the origin, and its diagonal blocks are even holomorphic there, while the others could be singular there. Direct estimates of (1.15) (p. 13) show that the blocks of $T(z)$ below the diagonal cannot have an essential singularity at the origin. Moreover, the definition of $T(z)$ implies

$$z\, T'(z) = A(z)\, T(z) - T(z)\, B(z), \tag{2.15}$$

with $B(z) = K + z^K M\, z^{-K}$. Since $\kappa_1, \ldots, \kappa_\mu$ are weakly increasing and M is lower triangularly blocked, we conclude that $B(z)$ is holomorphic at the origin, and $B(0)$ and $A(0)$ have the same diagonal blocks. For the blocks $T_{kj}(z)$, $1 \le j < k \le \nu$, we obtain from (2.15) that

$$z\, T'_{kj}(z) = A_{kk}(z)\, T_{kj}(z) - T_{kj}(z)\, B_{jj}(z) + R_{kj}(z), \tag{2.16}$$

with $R_{kj}(z)$ depending only upon such blocks of $T(z)$ that are closer to, or on, the diagonal. Assume that we have shown $R_{kj}(z)$ to be holomorphic at the origin, which is correct for $k - j = 1$, since then $R_{kj}(z)$ involves only diagonal blocks of $T(z)$. Then, let $m \ge 0$ be the pole order of $T_{kj}(z)$ and $T^{(kj)}_{-m}$ denote the corresponding coefficient in its Laurent expansion. If m were positive, then (2.16) would imply $[mI + A_{kk}(0)]\, T^{(kj)}_{-m} = T^{(kj)}_{-m}\, A_{jj}(0)$. This, however, would imply $T^{(kj)}_{-m} = 0$, owing to our assumption on the eigenvalues of the diagonal blocks. Hence $m = 0$ follows; i.e., $T_{kj}(z)$ is holomorphic at the origin. Therefore, by induction with respect to $k - j$ we find that all blocks of $T(z)$ must be holomorphic at the origin. □

Together, the above two propositions clarify the structure of fundamental solutions in case of general spectrum: First, we use a constant transformation to bring A_0 into Jordan form, with eigenvalues ordered appropriately. Then, we compute an upper-triangularly blocked analytic transformation, such that the transformed equation is lower triangularly blocked. Finally, we find a lower triangularly blocked analytic transformation for which the transformed equation has an explicit solution $z^K\, z^M$. We state this main result of the current section in the following theorem, before we add a remark concerning a more efficient way of computing this fundamental solution.

Theorem 6 *A system* (2.1) *with general spectrum has a fundamental solution of the form*

$$X(z) = T(z)\, z^K\, z^M, \tag{2.17}$$

where

- $T(z)$ *is an analytic transformation,*
- M *is constant, lower triangularly blocked of some type* $(\nu_1, \ldots, \nu_\mu)$*,*
- *the* k*th diagonal block of* M *has exactly one eigenvalue* m_k *with real part in the half-open interval* $[0, 1)$*,*
- $K = \operatorname{diag}[\kappa_1 I_{\nu_1}, \ldots, \kappa_\mu I_{\nu_\mu}]$*,* $\kappa_j \in \mathbb{Z}$*, weakly increasing and so that* $m_k + \kappa_k$ *is an eigenvalue of* A_0 *with algebraic multiplicity* ν_k*.*

It is worth pointing out that the above result does not coincide with Theorem 5 (p. 19) in case of a good spectrum, since the eigenvalues of A_0 need not have real parts in $[0, 1)$. Nonetheless, the theorem clarifies the structure of fundamental solutions of (2.1) with general spectrum, and the propositions in principle allow the computation of the monodromy matrix M, the diagonal matrix K, and any finite number of coefficients of $T(z)$. For a more effective computation, one may use the following procedure from *Gantmacher* [105], which also provides another proof of Theorem 6:

Remark 1: Let (2.1) and an analytic transformation $T(z) = \sum_0^\infty T_n\, z^n$ be arbitrarily given, and expand $B(z) = \sum_0^\infty B_n\, z^n$. Then (2.13) is equivalent to

$$T_n\,(B_0 + nI) - A_0\, T_n = \sum_{m=0}^{n-1} (A_{n-m}\, T_m - T_m\, B_{n-m}), \tag{2.18}$$

for $n \geq 0$. We assume that $A_0 = \operatorname{diag}[A_{11}, \ldots, A_{\mu\mu}]$ is diagonally blocked as in Proposition 3 (p. 30), which can always be brought about by a constant transformation, and take $T_0 = I$, $B_0 = A_0$. Blocking $T_n = [T_n^{(jk)}]$, and similarly A_n, B_n, according to the block structure of A_0, the equations (2.18) for $n \geq 1$ are of the form

$$T_n^{(jk)}\,(A_{kk} + nI) - A_{jj}\, T_n^{(jk)} = A_n^{(jk)} - B_n^{(jk)} + \ldots$$

According to Lemma 24 (p. 212), whenever $A_{kk} + nI$ and A_{jj} do not have the same eigenvalue, these equations can be solved for $T_n^{(jk)}$ for whatever block $B_n^{(jk)}$ we have, and we then choose such blocks equal to vanish. In the other case we apply Lemma 25 (p. 213) to see that the equation becomes solvable if we pick $B_n^{(jk)}$ accordingly. This second case occurs only for finitely many values of n and for some $j > k$, and one can see that the transformed system so obtained has a fundamental solution $z^K z^M$, with K and M as described in the theorem. ◇

A definition frequently used in the literature is as follows: Given a system having an isolated singularity at some point z_0, we say that z_0 is *a regular-singular point*, if a fundamental solution $X(z) = S(z)\,(z - z_0)^M$ exists whose single-valued part $S(z)$ has at most a pole at z_0, or in other words: if $\|X(z)\|$ cannot grow faster than some negative power of $|z-z_0|$ as $z \to z_0$ in, e.g., $|\arg(z-z_0)| \le \pi$. Note that then the same statements hold for other fundamental solutions as well. Using this terminology, we may summarize the results obtained in this chapter as follows:

1. Every singularity of first kind is a regular-singular point.
2. In case of good spectrum, there is a unique fundamental solution of (2.1) of the form (2.2), and the coefficients S_n can be recursively computed from (2.3).
3. In case of general spectrum, there exists a fundamental solution of (2.1) of the form (2.17), and the matrices M and K and the coefficients T_n can be computed as described in Remark 1.

So we have given quite satisfactory solutions to the problems stated at the beginning of this chapter. It is worthwhile to emphasize that the converse of statement 1 does not hold, as follows from Exercise 1. To find an effective algorithmic procedure for checking whether a system has a regular-singular point at z_0 is not a trivial matter and has attracted the attention of researchers for quite some time. In principle, every procedure finding the so-called *formal fundamental solution* that we shall define later can also serve as a way of determining the nature of the singularity. So many papers concerned with effective calculation of formal solutions fall into this category as well. From the more recent work in this direction, we mention, e.g, *Turrittin* [268], *Moser* [195], *Lutz* [171–173], *Deligne*[84], *Jurkat and Lutz* [145], *Harris* [114], *Wagenführer* [276, 277], and *Dietrich* [85–87].

In particular, the computer algebra packages mentioned in Section 13.5 may be used very effectively to check whether a singularity of positive Poincaré rank is regular-singular. For scalar equations, however, we shall obtain a very easy criterion for regular-singular points. So in this context, scalar equations are much “better behaved” than systems.

Exercises:

1. For an arbitrary constant M and a diagonal matrix K of integer diagonal entries, let $A(z) = K + z^K M\, z^{-K}$. Show that the origin is a regular-singular point of $zx' = A(z)\,x$, but in general a singularity of the second kind.
2. Let a confluent hypergeometric system (2.5) in dimension $\nu = 2$, with $B = \operatorname{diag}[0, 1]$ and A as in (2.7), be given. Use the procedure outlined in Remark 1 to find explicitly the matrices M and K, and recursion

equations for the entries of the matrices T_n, for a fundamental solution of the form (2.17).

2.5 Scalar Higher-Order Equations

Consider a νth-order linear ODE (1.6) (p. 4), with coefficients $a_k(z)$ holomorphic in some punctured disc $R(0,\rho)$ about the origin. As for systems, we say that the origin is a *regular-singular point* of (1.6), if all solutions do not grow faster than some inverse powers of $|z|$, as $z \to 0$ in $|\arg z| \le \pi$. Moreover, we call the origin a *singularity of the first kind* for (1.6), if $a_k(z)$ has at most a pole of kth order there, for $1 \le k \le \nu$. To see how this definition relates to the system case, see Exercise 1.

In contrast to the case of systems, for scalar equations regular-singular points and singularities of the first kind are the same:

Theorem 7 *The origin is a regular-singular point for* (1.6) *if and only if it is a singularity of the first kind.*

Proof: First, assume that (1.6) has a singularity of the first kind at the origin. Then the equivalent system constructed in Exercise 1 has a singularity of the first kind as well, hence a regular-singular point according to Theorem 6. This in turn implies a regular-singular point for (1.6).

Conversely, assume that (1.6) has a regular-singular point at the origin. From Exercise 9 on p. 10, we conclude existence of a solution of the form $y(z) = s(z)\, z^{\mu}$, with $\mu \in \mathbb{C}$ and $s(z)$ single-valued at the origin. Owing to the assumption of a regular-singular point, $s(z)$ cannot have an essential singularity at the origin. Redefining μ mod 1, we therefore may assume

$$y(z) = \sum_{n=0}^{\infty} y_n\, z^{n+\mu}, \quad |z| < \rho,$$

with $y_0 \neq 0$. As for systems, we call such a solution $y(z)$ *a Floquet solution of* (1.6). Making ρ smaller if necessary, we may also assume that $y(z)$ does not vanish on $R(0,\rho)$. We now complete the proof by induction with respect to ν: For $\nu = 1$, we obtain from (1.6) that $z\, a_1(z) = zy'(z)/y(z)$, and on the right one can cancel z^{μ}. Thus $z\, a_1(z)$ is the quotient of two holomorphic functions, with a nonzero denominator, hence holomorphic at the origin. So for $\nu = 1$ we completed the proof. For larger ν, we substitute $y = y(z)\tilde{y}$ into (1.6), use that $y(z)$ is a solution to cancel two terms, and then divide by $y(z)$, to obtain (setting $a_0(z) \equiv -1$):

$$\tilde{y}^{(\nu)} = \sum_{j=1}^{\nu-1} \tilde{y}^{(j)} \sum_{k=j}^{\nu} \binom{k}{j} \frac{y^{(k-j)}(z)}{y(z)}\, a_{\nu-k}(z).$$

This is an ODE for $\tilde{y}\,'$, and its order is $\nu-1$. Observe that the factor z^μ can be canceled from $y^{(k-j)}(z)/y(z)$, which then becomes single-valued at the origin, with a pole of order $j-k$. So the coefficients of this new equation are holomorphic in $R(0,\rho)$. By assumption, no solution y (hence: no $\tilde{y}$, or its derivative) can grow faster than powers of z, so by induction hypothesis, the new ODE has a singularity of the first kind at the origin. Thus we conclude that the coefficient of $\tilde{y}^{(j)}$ can at most have a pole of order $\nu-j$ at the origin. This information can be used to conclude by induction that each $a_{\nu-k}(z)$ can at most have a pole of order $\nu-k$. □

According to Exercise 3, one can always recursively compute the coefficients for at least one Floquet solution of (1.6). The proof of the previous theorem then shows that other solutions can, in principle, be obtained through an equation of order $\nu-1$. This, however, is not a very efficient procedure.

The Floquet exponents μ are roots of the so-called *indicial equation* introduced in Exercise 2. A fundamental solution of (1.6), aside from Floquet solutions, consists of so-called *logarithmic solutions* of the form

$$y(z) = z^\mu \sum_{k=0}^{j} s_k(z)\,[\log z]^k,$$

where μ is one of the Floquet exponents, j is strictly smaller than its multiplicity, and the $s_k(z)$ are holomorphic at the origin. An elegant method to compute finitely many power series coefficients of these functions goes under the name *Frobenius' method*. In the exercises below we have described an essential part of this method; for the cases not included there, see [82]. In any case, for $\nu \geq 3$, one will only in exceptional cases succeed in finding *all* power series coefficients of the functions $s_k(z)$ in closed form.

Exercises: For the following exercises, let a νth-order linear ODE (1.6) in $G = R(0,\rho)$, $\rho > 0$, be given.

1. Defining $x = [y, zy', \ldots, z^{\nu-1}y^{(\nu-1)}]^T$, and

$$A(z) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \nu-2 & 1 \\ b_\nu(z) & b_{\nu-1}(z) & b_{\nu-2}(z) & \ldots & b_2(z) & b_1(z)+\nu-1 \end{bmatrix},$$

with $b_k(z) = z^k a_k(z)$, show that $y(z)$ solves (1.6) if and only if the corresponding $x(z)$ is a solution of (2.1) (in particular, show that a solution vector $x(z)$ of (2.1) always is of the above form).

2. Assume (1.6) to have a singularity of the first kind at the origin, i.e.,

$$b_k(z) = z^k a_k(z) = \sum_{n \geq 0} b_n^{(k)} z^n, \quad |z| < \rho, \ 1 \leq k \leq \nu.$$

Let $y(z) = \sum_n y_n \, z^{n+\mu}$, $y_0 \neq 0$, $\mu \in \mathbb{C}$, be a solution of (1.6). Show that then the Floquet exponent μ satisfies the *indicial equation*

$$[\mu]_\nu = \sum_{k=1}^{\nu} b_0^{(k)} \, [\mu]_{\nu-k},$$

with $[\mu]_0 = 1$, $[\mu]_k = \mu(\mu-1) \cdot \ldots \cdot (\mu - k + 1)$, for $k \in \mathbb{N}$.

3. For $b_k(z)$ and $y(z)$ as above, find the equations that the coefficients y_n have to satisfy so that $y(z)$ is a solution of (1.6). Show that for at least one solution μ of the indicial equation one can recursively compute the coefficients from these equations. In this way, we obtain a constructive proof for existence of at least one Floquet solution.

4. For $b_k(z)$ as above, let w be a complex variable.

   (a) With $p_0(w) = [w]_\nu - \sum_{k=1}^{\nu} b_0^{(k)} [w]_k$, show that the inhomogeneous equation

   $$z^\nu y^{(\nu)} - \sum_{k=1}^{\nu} z^{\nu-k} b_k(z) \, y^{(\nu-k)} = p_0(w) \, z^w$$

   has a unique solution $y(z; w) = \sum_0^\infty y_n(w) \, z^{n+w}$, with $y_0(w) \equiv 1$ and coefficients $y_n(w)$, $n \geq 1$, which are rational functions of w. Find the possible poles of $p_n(w)$ in terms of the roots of $p_0(w)$.

   (b) Let μ be a root of $p_0(w)$, but so that $p_0(\mu + j) \neq 0$ for $j \in \mathbb{N}$. Conclude that then $y(z; \mu)$ is a Floquet solution of (1.6). Show that for at least one root of $p_0(w)$ this assumption holds.

   (c) Let μ, as above, be a root of multiplicity at least $k \geq 2$. Show that then (1.6) has a solution of the form

   $$\sum_{j=0}^{k-1} \binom{k-1}{j} [\log z]^{k-1-j} \sum_{n=0}^{\infty} y_n^{(j)}(\mu) \, z^{n+\mu}, \quad |z| < \rho.$$

   Show that all solutions so obtained are linearly independent.

5. For the hypergeometric, resp. Bessel's, equation, find the cases where the indicial equation has a double root. Use the previous exercise to compute a fundamental solution for these cases.

3

Highest-Level Formal Solutions

In this and later chapters, we are concerned with systems having *a singularity of the second kind* at some point z_0. As in the case of singularities of the first kind, we may without loss of generality assume that z_0 is any preassigned point in $\mathbb{C} \cup \{\infty\}$. Both for historical reasons as well as notational convenience it is customary to choose $z_0 = \infty$ here. Hence we shall be dealing with systems of the form

$$z\, x' = A(z)\, x, \quad A(z) = z^r \sum_{n=0}^{\infty} A_n\, z^{-n}, \qquad |z| > \rho, \tag{3.1}$$

with Poincaré rank $r \geq 1$. Observe that we usually assume the *leading term* A_0 to be nonzero, but sometimes we may apply certain transformations, producing a new system with vanishing leading term. In this case we may say *that we have lowered the Poincaré rank of the system.*

In principle, we should like to solve the same three problems posed at the beginning of the previous chapter, but we shall see that things are considerably more complicated here: We know that fundamental solutions of (3.1) are of the form $X(z) = S(z)\, z^M$, with a single-valued matrix $S(z)$ that in general will have an essential singularity at infinity. Expanding $S(z) = \sum_{-\infty}^{\infty} S_n z^{-n}$, inserting into (3.1), and comparing coefficients gives

$$S_{n-r}(M - (n-r)I) = \sum_{m=0}^{\infty} A_m S_{n-m}, \quad n \in \mathbb{Z}.$$

This is a homogeneous system of infinitely many equations in infinitely many unknowns; and the also unknown matrix M may be regarded as a *matrix eigenvalue* for this system. In the case of a singularity of the first kind we have seen that $S(z)$ cannot have an essential singular point at infinity; hence $S_n = 0$ for small $n \in \mathbb{Z}$. For singularities of the second kind, however, this is no longer true. Therefore, although we know that for some M this system has a solution $(S_n)_{-\infty}^{\infty}$ for which $S(z)$ converges and $\det S(z) \neq 0$ for sufficiently large $|z|$, to find M and (S_n) is much more difficult. There is a theory of infinite determinants (see *von Koch* [153]) that might be applied here. We shall not do this, however, because even if we had computed M and the coefficients S_n, we would not obtain any detailed information on the behavior of solutions for $z \to \infty$. Instead, we will show existence of certain transformations that will block-diagonalize the system (3.1). Hence in principle we will reduce the task of studying the nature of solutions of (3.1) near the point infinity to the same problem for *simpler, i.e., smaller, systems.* However, some of the transformations will be *formal* in the sense that they look like analytic transformations, but the radius of convergence of the power series will in general be equal to zero. So at first glance, the usefulness of these transformations is questionable, but in several later chapters we shall show that they can nonetheless be given a clear meaning.

Many of the books listed in Chapter 1 also cover the theory of singularities of the second kind. In addition, we mention the survey articles by *Brjuno* [74], *Kimura* [151], *Malgrange* [182, 183], *Hukuhara* [131, 132], *Bertrand* [51], and *Varadarajan* [274, 275]. For presentations using more algebraic tools, see *Gerard and Levelt* [106], *Levelt* [167], or *Babbitt and Varadarajan* [3].

3.1 Formal Transformations

Since we are now working in a neighborhood of infinity, we have to adjust the notion of analytic transformations accordingly. Moreover, we shall have reason to use other types of transformations as well, and we start by listing such transformations $x = T(z)\tilde{x}$ employed in this chapter:

1. If $T(z) = \sum_0^\infty T_n\, z^{-n}$ has positive radius of convergence and $\det T_0 \neq 0$, then we shall call $T(z)$, or more precisely the change of the dependent variable $x = T(z)\tilde{x}$, an *analytic transformation* (at the point infinity). This coincides with the definition in Section 2.4, up to the change of variable $z \mapsto 1/z$. If only finitely many coefficients T_n are different from zero, we say that the transformation *terminates.*

2. If $\hat{T}(z) = \sum_0^\infty T_n\, z^{-n}$, with T_0 as above, but the radius of convergence of the power series possibly equals zero, we speak of a *formal analytic transformation.* If in addition for some $s \geq 0$ we can find constants

$c, K > 0$ so that

$$\|T_n\| \le c\,K^n \Gamma(1+sn), \quad n \ge 0, \tag{3.2}$$

we shall say that $\hat{T}(z)$ is a formal analytic transformation *of Gevrey order s*. Note that being of Gevrey order $s = 0$ is the same as convergence of $\hat{T}(z)$ for sufficiently large $|z|$.

3. Transformations of the form $T(z) = \operatorname{diag}[z^{p_1/q}, \ldots, z^{p_\nu/q}]$, with $p_k \in \mathbb{Z}$, $q \in \mathbb{N}$, will be called *shearing transformations.* If $q = 1$, we say that $T(z)$ is an *unramified* shearing transformation.

4. For $T(z) = \exp[q(z)]I$, with $q(z)$ a scalar polynomial in z, we speak of a *scalar exponential shift.* Note that such a transformation is holomorphic at the origin, but is essentially singular at infinity. However, since $T(z)$ commutes with $A(z)$, the transformed system has coefficient matrix $A(z) - zq'(z)I$ and hence is again meromorphic at infinity.

5. A transformation of the form $T(z) = \sum_{n=-n_0}^{\infty} T_n z^{-n}$, with $n_0 \in \mathbb{Z}$, will be called a *meromorphic* transformation if the series converges for sufficiently large $|z|$, and if in addition the determinant of $T(z)$ is not the zero series. Then, $T^{-1}(z)$ exists and is again of the same form. If the radius of convergence of the series is unknown, and in particular may be equal to zero, we speak of a *formal meromorphic transformation*, and in this case write $\hat{T}(z)$ instead of $T(z)$. If the coefficients T_n satisfy (3.2), we call $\hat{T}(z)$ a *formal meromorphic transformation of Gevrey order s.*

6. A transformation $T(z)$ will be called *q-analytic, resp. q-meromorphic*, with $q \in \mathbb{N}$, if $T(z^q)$ is an analytic, resp. meromorphic, transformation, or in other words, if $T(z) = \sum_{n=-n_0}^{\infty} T_n z^{-n/q}$, $|z| > \rho$, and $\det T(z)$ is not the zero series. Analogously, we define *formal q*-analytic resp. q-meromorphic transformations $\hat{T}(z)$. Finally, we say that $\hat{T}(z)$ is *formal of Gevrey order s* if $\hat{T}(z^q)$ is formal of Gevrey order s/q, i.e., if the coefficients T_n satisfy (3.2) with s/q in place of s. To see that this is a natural terminology, check what happens to the Gevrey order of an analytic resp. meromorphic transformation under a change of variable $z = w^q$.

Note that all the above transformations have inverses of the same type: For formal analytic transformations of Gevrey order s compare Exercise 3, and for meromorphic ones use Proposition 5 (p. 40); observe, however, that the inverse of a terminating transformation will, in general, not terminate. Proceeding formally, the change of variable $x = T(z)\,\tilde{x}$ transforms (3.1) into the *transformed system* $z\,\tilde{x}' = B(z)\,\tilde{x}$, with

$$z\,T'(z) = A(z)\,T(z) - T(z)\,B(z).$$

However, except for analytic transformations, the transformed system will in general not be of the same form as (3.1): In case of a formal analytic transformation the coefficient matrix will be given as a series of the form (3.1), but the radius of convergence of this series will in general be equal to zero. Systems where this happens, or to be precise: *where this may happen*, will be named *formal systems* and denoted as

$$z\,x' = \hat{A}(z)\,x, \quad \hat{A}(z) = z^r \sum_{n=0}^{\infty} A_n\, z^{-n}. \tag{3.3}$$

In particular, if for sufficiently large $c, K > 0$ we have

$$\|A_n\| \le c\,K^n\,\Gamma(1+sn), \quad n \ge 0, \tag{3.4}$$

then we shall call (3.3) a *formal system of Gevrey order* s. Applying a formal analytic or meromorphic transformation $\hat{T}(z)$ to (3.3) and denoting the resulting formal system by $z\tilde{x} = \hat{B}(z)\,\tilde{x}$, the coefficient matrices are related by the purely formal identity

$$z\,\hat{T}'(z) = \hat{A}(z)\,\hat{T}(z) - \hat{T}(z)\,\hat{B}(z). \tag{3.5}$$

Using Exercise 2, we see that a formal analytic transformation of Gevrey order s transforms a system that is formal of Gevrey order s and of Poincaré rank r into another such system. Shearing transformations, or q-analytic transformations, take (3.1) or (3.3) into a system that, after a change of the independent variable as in Exercise 5, is of the same form, but perhaps of larger rank. Scalar exponential shifts also increase the Poincaré rank of the system, except when the degree of $q(z)$ is at most r.

For meromorphic transformations, we prove the following well-known result, showing that they are a combination of analytic ones and a shearing transformation:

Proposition 5 *Every formal meromorphic transformation $\hat{T}(z)$ can be factored as*

$$\hat{T}(z) = \hat{T}_1(z)\,\mathrm{diag}\,[z^{k_1}, \ldots, z^{k_\nu}]\,\hat{T}_2(z),$$

where $\hat{T}_1(z), \hat{T}_2(z)$ are formal analytic transformations and $k_j \in \mathbb{Z}$, $k_1 \le k_2 \le \ldots \le k_\nu$. When $\hat{T}(z)$ is of Gevrey order $s \ge 0$, then both $\hat{T}_j(z)$ can be chosen to be of Gevrey order s as well.

Proof: We proceed by induction with respect to ν: For $\nu = 1$, the statement obviously holds with $\hat{T}_1(z) \equiv 1$ and $K = k = \deg \hat{T}(z)$, where $\deg \hat{T}(z)$ denotes the highest power of z occurring in the series $\hat{T}(z)$. For $\nu \ge 2$, let $\hat{T}(z) = [\hat{t}_{jk}(z)]$ and take $k_\nu = \deg \hat{T}(z) = \max_{j,k} \deg \hat{t}_{jk}(z)$. Interchanging rows and columns, we can arrange that $k_\nu = \deg \hat{t}_{\nu\nu}(z)$. Adding suitable multiples of the last row/column to previous ones, the factors used being formal power series in z^{-1}, and then dividing the last row by a power series with nonzero constant term, we can arrange that $\hat{t}_{\nu\nu}(z) \equiv z^{k_\nu}$, $\hat{t}_{\nu j}(z) = \hat{t}_{j\nu}(z) \equiv 0$, $1 \leq j \leq \nu - 1$. All the steps used to obtain this form can be interpreted as multiplication from the left- or right-hand side by formal analytic transformations, which are of Gevrey order $s \geq 0$ if $\hat{T}(z)$ is formal of Gevrey order s. The induction hypothesis then completes the proof. □

While the usefulness of formal transformations is not clear offhand, the other types of transformations take (3.1) into an *equivalent* system in the sense that it suffices to compute a fundamental solution of either one of the two equations, since then one for the other system is obtained through the transformation. We shall see in Chapter 8 that the same applies to formal transformations of Gevrey order s as well, since we shall give a holomorphic interpretation to the formal series by which formal transformations are defined. In this chapter, however, we shall take a *formal approach*, meaning that we most of the time disregard the question of convergence of formal power series occurring in calculations, but we will always verify estimates of the form (3.2) resp. (3.4).

Exercises:

1. Use the *Beta Integral* (p. 229) to show $\Gamma(1+x)\Gamma(1+y)/\Gamma(1+x+y) = x\,B(x, 1+y) \leq 1$ for $x, y > 0$.

2. If $\hat{T}_1(z), \hat{T}_2(z)$ are formal analytic transformations of Gevrey order $s \geq 0$, show the same for $\hat{T}_1(z)\,\hat{T}_2(z)$.

3. For a formal analytic transformation $\hat{T}(z)$ of Gevrey order s, show that $\hat{T}^{-1}(z)$ is of Gevrey order s, too.

4. Show that every formal meromorphic transformation can be factored as $T(z)\,\hat{T}(z)$, with a formal analytic transformation $\hat{T}(z)$ and a terminating meromorphic transformation $T(z)$. Conclude that in Proposition 5 we may take $\hat{T}_1(z)$ to converge.

5. Show that the system equivalent to (3.3) by means of the change of variable $z = w^q$, $q \in \mathbb{N}$, has Poincaré rank qr.

6. Show that for $s > 0$ the following statements both are equivalent to (3.2), possibly for different values of c, K:

   (a) $\|T_n\| \leq c\,K^n\,n^{sn}$, $n \geq 1$. (b) $\limsup \sqrt[n]{\|T_n\|/\Gamma(1+sn)} < \infty$.

3.2 The Splitting Lemma

Roughly speaking, finding a *formal fundamental solution* of (3.1) will be equivalent to finding a formal q-meromorphic transformation $\hat{T}(z)$ so that the transformed system, after a change of variable $z = w^q$, is elementary in the sense of Section 1.6. A fundamental solution $G(z)$ of the transformed system can then be easily computed. The object $\hat{X}(z) = \hat{T}(z)\, G(z)$ formally satisfies (3.1), and classically these formal fundamental solutions have been the starting point of the so-called *asymptotic theory.* In the light of recent results, it will be more natural to base these investigations on a *formal solution of highest level.* This essentially will be a formal q-meromorphic transformation $\hat{T}(z)$ of a certain minimal Gevrey order, block-diagonalizing the system (3.1). The diagonal blocks of the transformed system will, in general, not be elementary. However, since they are of smaller dimensions than the original system, we shall be able to obtain significant results simply *by induction with respect to the dimension of the system.*

When the leading term A_0 of (3.1) has several distinct eigenvalues, existence of such a transformation, with $q = 1$, follows from a classical result:

Lemma 3 (Splitting Lemma) *Let* (3.3) *be a formal system of Gevrey order s, and assume that $A_0 = \operatorname{diag}[A_0^{(11)}, A_0^{(22)}]$, such that the two diagonal blocks have disjoint spectra. Then there exists a unique formal analytic transformation of Gevrey order $\tilde{s} = \max\{s, 1/r\}$ of the form*

$$\hat{T}(z) = \begin{bmatrix} I & \hat{T}_{12}(z) \\ \hat{T}_{21}(z) & I \end{bmatrix}, \quad \hat{T}_{jk}(z) = \sum_1^\infty T_n^{(jk)}\, z^{-n},$$

such that the transformed formal system is diagonally blocked, with each of the two diagonal blocks being a formal system of Gevrey order $\tilde{s}$.

Proof: Blocking

$$\hat{A}(z) = \begin{bmatrix} \hat{A}_{11}(z) & \hat{A}_{12}(z) \\ \hat{A}_{21}(z) & \hat{A}_{22}(z) \end{bmatrix}, \quad \hat{B}(z) = \begin{bmatrix} \hat{B}_{11}(z) & 0 \\ 0 & \hat{B}_{22}(z) \end{bmatrix},$$

and inserting into (3.5) leads to

$$\hat{B}_{22}(z) = \hat{A}_{22}(z) + \hat{A}_{21}(z)\, \hat{T}_{12}(z), \tag{3.6}$$

$$\begin{aligned} z\hat{T}_{12}'(z) \quad = \quad & \hat{A}_{12}(z) + \hat{A}_{11}(z)\, \hat{T}_{12}(z) - \hat{T}_{12}(z)\, \hat{A}_{22}(z) \\ & -\hat{T}_{12}(z)\, \hat{A}_{21}(z)\, \hat{T}_{12}(z), \end{aligned} \tag{3.7}$$

plus two other equations with indices $1, 2$ permuted that are omitted here but can be treated in quite the same way. Inserting power series expansions and comparing coefficients implies

$$\begin{aligned} T_n^{(12)} A_0^{(22)} - A_0^{(11)} T_n^{(12)} &= \sum_{m=1}^{n-1} (A_{n-m}^{(11)} T_m^{(12)} - T_m^{(12)} A_{n-m}^{(22)}) \\ &\quad - \sum_{\mu=1}^{n-2} T_\mu^{(12)} \sum_{m=1}^{n-\mu-1} A_{n-m-\mu}^{(21)} T_m^{(12)} \\ &\quad + A_n^{(12)} + (n-r) T_{n-r}^{(12)}, \end{aligned} \tag{3.8}$$

for $n \geq 1$, interpreting $T_{n-r}^{(12)} = 0$ for $n \leq r$. From these formulas we can uniquely compute the coefficients $T_n^{(12)}$. So we are left to estimate the coefficients $T_n^{(12)}$, and this can be done as follows (also compare the proof of Lemma 2 (p. 28)):

It suffices to consider the case $s \geq 1/r$, so that $\tilde{s} = s$. By assumption we have (3.4). Taking $t_n = \|T_n^{(12)}\| K^{-n} / \Gamma(1+sn)$, we conclude from (3.8), for sufficiently large $\tilde{c} > 0$:

$$\begin{aligned} t_n \quad \leq \quad & \tilde{c} \Big[1 + 2 \sum_{m=1}^{n-1} \frac{\Gamma(1+s(n-m))\Gamma(1+sm)}{\Gamma(1+sn)} t_m \\ & + \sum_{\mu=1}^{n-1} t_\mu \sum_{m=1}^{n-\mu-1} \frac{\Gamma(1+s\mu)\Gamma(1+s(n-\mu-m))\Gamma(1+sm)}{\Gamma(1+sn)} t_m \\ & + K^{-r} \frac{(n-r)\Gamma(1+s(n-r))}{\Gamma(1+sn)} t_{n-r} \Big], \quad n \geq 1. \end{aligned}$$

According to Exercise 1 on p. 41, both quotients of Gamma functions inside the two sums can be estimated by 1, and by taking K larger we may also assume the term in front of t_{n-r} to be bounded by 1. Finally, by allowing t_n to become larger, we may assume equality in the above estimate, and thus get $t_n = \tilde{c}\,[1 + t_{n-r} + \sum_{\mu=1}^{n-1} t_\mu (2 + \sum_{m=1}^{n-\mu-1} t_m)]$, for $n \geq 1$. Defining $f(z) = \sum_1^\infty t_n z^n$, we conclude formally

$$(1-z) f(z) = z\,\tilde{c}\,[1 + (2 + z^{r-1}(1-z))\, f(z) + f^2(z)].$$

This quadratic equation has exactly one solution that is holomorphic at the origin and vanishes there, while the other solution has a pole. This shows convergence of $\sum_1^\infty t_n z^n$, hence $t_n \leq \tilde{K}^n$. Thus we obtain an estimate of the desired type for $\|T_n^{(12)}\|$. In the same fashion one can estimate $\|T_n^{(21)}\|$, hence $T(z)$ is of Gevrey order s. Moreover, from (3.6) we conclude, using Exercise 2 on p. 41, that both diagonal blocks of the transformed system are of Gevrey order s. □

Applying the above Lemma repeatedly and using Exercise 2 on p. 41, we obtain an analogous result when A_0 has more than two blocks:

Theorem 8 *Let* (3.3) *be a formal system of Gevrey order* $s \geq 0$*, and assume that* $A_0 = \operatorname{diag}[A_0^{(11)}, \ldots A_0^{(\mu\mu)}]$*, such that the kth diagonal block has exactly one eigenvalue* λ_k*, and let* $\lambda_1, \ldots, \lambda_\mu$ *be distinct. Then there exists a unique formal analytic transformation of Gevrey order* $\tilde{s} = \max\{s, 1/r\}$*, with diagonal blocks all equal to* I *and the off-diagonal ones having zero constant terms, such that the transformed formal system is diagonally blocked, with each of the diagonal blocks being formal of Gevrey order* $\tilde{s}$*.*

Remark 2: It follows from the form of the formal analytic transformation that the diagonal blocks of the transformed system have leading terms with exactly one eigenvalue. Assuming that this was already the case for (3.3), we can use the scalar exponential shift $x = \exp[\lambda z^r/r]\ \tilde{x}$ to produce a system with nilpotent matrix A_0. Such systems will be investigated in the following section. ◇

Remark 3: Suppose that A_0 has ν distinct eigenvalues; this situation usually is referred to as the *distinct eigenvalue case.* Then the above theorem shows existence of a formal analytic transformation for which the transformed system is diagonal, i.e., *decoupled into* ν *one-dimensional systems.* These can easily be solved, and in this way one can compute a formal fundamental solution of (3.3). For details, compare Exercise 4. ◇

Exercises:

1. For $r = 1$, $\nu = 2$ and

$$A_0 = \operatorname{diag}[0, 1],\ A_1 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},\ A_n = 0,\ n \geq 2,$$

compute $T_n^{(12)}$ given by (3.8) and find its rate of growth, in dependence upon a and b, as $n \to \infty$.

2. Assume that a formal system (3.3) is given, whose coefficients A_n all commute with one another (this is satisfied for $\nu = 1$, or whenever all A_n are diagonal). Show that $B(z) = A_r \log z + \sum_{n \neq r} A_n z^{r-n}/(r-n)$ commutes with $A(z)$. In case the series in (3.3) converges for $|z| > \rho$, conclude that (3.3) is elementary in the sense of Section 1.6.

3. Under the assumptions of the previous exercise, show that (3.3), at least formally, i.e., disregarding the question of convergence of the series involved, has the fundamental solution $\hat{X}(z) = \hat{T}(z)\, z^\Lambda\, e^{Q(z)}$, with $Q(z) = \sum_{n=0}^{r-1} A_n z^{r-n}/(r-n)$, $\Lambda = A_r$, and

$$\hat{T}(z) = I + \sum_1^\infty T_n z^{-n} = \exp\big[\sum_{n>r} A_n z^{r-n}/(r-n)\big].$$

4. In case the leading term A_0 has all distinct eigenvalues, show that (3.3) has a *formal fundamental solution* of the form
$$\hat{X}(z) = \hat{T}(z)\, z^{\Lambda}\, \mathrm{e}^{Q(z)},$$
with a formal analytic transformation $\hat{T}(z)$, being of Gevrey order $\tilde{s} = \max\{s, 1/r\}$ in case (3.3) is formal of Gevrey order s, a constant diagonal matrix Λ, and a diagonal matrix $Q(z)$ with polynomials along the diagonal satisfying $Q(0) = 0$. Explicitly describe an algorithm for computing the matrices Λ and $Q(z)$ and any finite number of coefficients of $\hat{T}(z)$.

3.3 Nilpotent Leading Term

In this section, we are concerned with formal systems (3.3) whose leading term only has one eigenvalue, so that the Splitting Lemma does not apply. A scalar exponential shift then reduces such a system to one with nilpotent leading term, and such a transformation commutes with every other one we are using here. Hence it is without loss of generality when, for notational convenience, we restrict our discussion to systems (3.3) having a nilpotent leading term $A_0 \neq 0$. We may also assume A_0 in Jordan canonical form

$$A_0 = \operatorname{diag}[N_1, \ldots, N_\mu], \tag{3.9}$$

with nilpotent Jordan blocks of weakly decreasing sizes $s_1 \geq \ldots \geq s_\mu$ – if this were not so, we could apply a constant transformation. In what follows, we shall block $\hat{A}(z) = [\hat{A}_{jk}(z)]$ according to the block structure of the matrix A_0, and similarly for transformations resp. corresponding transformed equations. Our goal is to simplify the given system, in a sense to be made clear later, using finitely many transformations that are either terminating analytic ones, or shearing transformations, or scalar exponential shifts. Owing to the nature of the transformations we see that in case the system we begin with is convergent, then the one we obtain in the end converges, too. Hence, divergent transformations or systems will only occur when applying the Splitting Lemma!

As a first step we shall try to find terminating analytic transformations, resp. unramified shearings, that leave the Poincaré rank r of (3.3) unchanged, but possibly increase the matrix rank of A_0 or, in general, leave the ranks of A_0^k fixed, for $1 \leq k \leq j-1$, and increase the rank of A_0^j. To have a way of expressing this, we shall use the following terminology:

Order Relation for Nilpotent Matrices

Given any two nilpotent matrices $N, \tilde{N} \in \mathbb{C}^{\nu \times \nu}$, we say that $\tilde{N}$ is superior to N, if for some $j \geq 1$ we have $\operatorname{rank} N^k = \operatorname{rank} \tilde{N}^k$ for $1 \leq k \leq j-1$, and $\operatorname{rank} N^j < \operatorname{rank} \tilde{N}^j$.

Note that the ranks of the powers determine the structure of a nilpotent Jordan matrix up to a permutation of its blocks, so if we restrict ourselves to nilpotent Jordan matrices whose Jordan blocks have decreasing sizes, then the above relation becomes a total order, and it is easily seen that the nilpotent matrix with rank equal to $\nu - 1$ is the maximal element for this order relation.

Classically, the case of nilpotent leading matrices has been treated by finding transformations *reducing the rank* of A_0, or if possible, even making $A_0 = 0$, hence lowering the Poincaré rank. Here we do the opposite, which at first glance appears unreasonable. However, we shall see that this may be considered as an example of what sometimes is called "worst case analysis," since in case of a maximal leading term we shall find that the system has a very particular structure.

Finding the transformations that will produce a superior leading term requires two steps, the first one being the following analogue to the Splitting Lemma; see, e.g., *Wasow* [281] for an equivalent version.

Lemma 4 *Let a formal system* (3.3) *with leading term as in* (3.9) *be given. Then for every $n_0 \in \mathbb{N}$ there exists an analytic transformation of the form $T(z) = I + [T_{jk}(z)]$, blocked according to the block structure of A_0, with matrices $T_{jk}(z) = \sum_1^{n_0} T_n^{(jk)} z^{-n}$, such that the coefficient matrix of the transformed system has the form $\hat{B}(z) = z^r A_0 + [\hat{B}_{jk}(z)]$, with $\hat{B}_{jk}(z) = z^r \sum_1^\infty B_n^{(jk)} z^{-n}$, and so that for $1 \le n \le n_0$ the coefficients $B_n^{(jk)}$*

- *have all zero rows except for the first one in case $1 \le j \le k \le \mu$,*
- *have all zero columns except for the last one in case $1 \le k < j \le \mu$.*

The transformation $T(z)$ is unique if we require for $1 \le n \le n_0$ that

- *all $T_n^{(jk)}$ have vanishing last row in case $1 \le j \le k \le \mu$,*
- *all $T_n^{(jk)}$ have vanishing first column in case $1 \le k < j \le \mu$.*

Proof: Insertion into (3.5) implies the following identities for the coefficients of the blocks of $\hat{A}(z), \hat{B}(z), T(z)$:

$$N_j\, T_n^{(jk)} - T_n^{(jk)}\, N_k = B_n^{(jk)} + R_n^{(jk)}, \qquad n \ge 1,\ 1 \le j, k \le \mu, \qquad (3.10)$$

where $R_n^{(jk)}$ only involves blocks of T_m, B_m with $m < n$. For $n \le n_0$, Lemma 25 (p. 213) implies existence of a unique matrix $B_n^{(jk)}$ with nonzero entries only in the first row, resp. last column, such that (3.10) has a solution $T_n^{(jk)}$. This solution is unique if we require its last row, resp. first column, to vanish. Which case applies depends on the shape of the block, i.e., upon its position on or above, resp. below, the block-diagonal. For $n \ge n_0 + 1$, we have $T_n^{(jk)} = 0$; hence take $B_n^{(jk)}$ so that (3.10) holds, thus completing the proof. □

Remark 4: We say that (3.3) (with A_0 as above and $\mu \geq 1$) is *normalized up to* z^{-n_0}, if all coefficients $A_n^{(jk)}$, for $1 \leq n \leq n_0$, have nonzero entries only in first row resp. last column (in case $j \leq k$ resp. $j > k$). If this is so for *some* n_0, we briefly call (3.3) normalized. If (3.3) is normalized up to z^{-n_0}, and in addition all $A_n^{(jk)}$ with $j \neq k$ vanish completely, for $1 \leq n \leq n_0$, then we say that (3.3) is *reduced up to* z^{-n_0}.

For later use, observe that if (3.3) is already normalized up to $z^{-\tilde{n}_0}$, then normalization up to z^{-n_0}, for $n_0 > \tilde{n}_0$, can be done by a transformation $T(z)$ with coefficients $T_n = 0$ for $1 \leq n \leq \tilde{n}_0$; hence the corresponding coefficients A_n remain unchanged. ◇

For normalized systems (3.3) with $\mu > 1$ we apply shearing transformations to produce another system with superior leading term:

Proposition 6 *For some $n_0 \in \mathbb{N}$, let* (3.3) *be a formal system with leading term as in* (3.9)*, normalized up to z^{-n_0}, and assume $\mu \geq 2$. Assume the existence of n_1, $1 \leq n_1 \leq n_0$, so that $\hat{A}_{k\mu}(z) = z^r \sum_{n \geq n_1} A_n^{(k\mu)} z^{-n}$, $1 \leq k \leq \mu - 1$, and not all $A_{n_1}^{(k\mu)}$ vanish. Then the unramified shearing transformation $T(z) = \operatorname{diag}[I_{s_1}, \ldots, I_{s_{\mu-1}}, z^{n_1} I_{s_\mu}]$ produces a system with a superior leading term.*

Proof: Abbreviate $C_k = A_{n_1}^{(k\mu)}$, $1 \leq k \leq \mu - 1$. Then, check that the shearing transformation produces a transformed system with leading term

$$B_0 = \begin{bmatrix} N_1 & 0 & \ldots & 0 & C_1 \\ 0 & N_2 & \ldots & 0 & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & N_{\mu-1} & C_{\mu-1} \\ 0 & 0 & \ldots & 0 & N_\mu \end{bmatrix}, \quad C_k = \begin{bmatrix} c_{k,1} & \ldots & c_{k,s_\mu} \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix}.$$

By assumption, $C_k \neq 0$ for at least one k. For $\ell \geq 1$, we find

$$B_0^\ell = \begin{bmatrix} N_1^\ell & 0 & \ldots & 0 & C_1^{(\ell)} \\ 0 & N_2^\ell & \ldots & 0 & C_2^{(\ell)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & N_{\mu-1}^\ell & C_{\mu-1}^{(\ell)} \\ 0 & 0 & \ldots & 0 & N_\mu^\ell \end{bmatrix},$$

with $C_k^{(\ell)} = N_k^{\ell-1} C_k + C_k^{(\ell-1)} N_\mu$, $\ell \geq 2$. This shows that the ℓth row of $C_k^{(\ell)}$ equals the first one of C_k, while the following ones vanish. From this we learn that those rows of B_0^ℓ with a one in a subdiagonal coincide with the corresponding rows of A_0^ℓ. Therefore, we have $\operatorname{rank} A_0^\ell \leq \operatorname{rank} B_0^\ell$, and equality holds if and only if the rows of $C_k^{(\ell)}$, for every $k = 1, \ldots, \mu - 1$, are linear combinations of the rows of N_μ^ℓ. Now, take $\ell = s_\mu$, the size of the last, i.e., the smallest, Jordan block of A_0. Then we have $N_\mu^\ell = 0$, but $C_k^{(\ell)} \neq 0$ for at least one k, hence inequality holds for the ranks, thus B_0 is superior to A_0. □

Suppose that we are given a formal system with a leading term as in (3.9), normalized up to z^{-n_0}. Proposition 6 then implies that either $\hat{A}_{k\mu}(z) = O(z^{r-n_0-1})$, $1 \leq k \leq \mu - 1$, or we have a shearing transformation producing a system with a superior leading term. Much more can be said, however:

Theorem 9 *Let* (3.3) *be a formal system with a leading term of the form* (3.9)*, normalized up to* z^{-n_0}*. Then we can find an unramified shearing transformation producing a system with a superior leading term, except when the coefficients* A_n*,* $1 \leq n \leq n_0$*, are all diagonally blocked in the block structure of* A_0*, i.e., when* (3.3) *is reduced up to* z^{-n_0}*.*

Proof: For $\mu = 1$ nothing remains to prove, so suppose otherwise. Applying Proposition 6 shows the existence of a shearing transformation as required, except when all A_n, $1 \leq n \leq n_0$, are lower triangularly blocked, not necessarily in the block structure of A_0, but the coarser one with two diagonal blocks, the second one of the same size as N_μ. It is not difficult to see that an analogue to Proposition 6, with a shearing transformation inverse to the one used there, produces a system with a superior leading term, except when the above-mentioned coefficients are diagonally blocked in the same coarser block structure. Repeating the same arguments for the first diagonal block then completes the proof. □

According to the above results, we are now left to deal with a formal system (3.3) with a leading term as in (3.9), and for some n_0 to be chosen later we may assume that (3.3) is reduced up to z^{-n_0}, i.e., for all $n = 1, \ldots, n_0$ and $1 \leq j, k \leq \mu$:

$$A_n^{(jj)} = \begin{bmatrix} a_n^{(j,1)} & a_n^{(j,2)} & \ldots & a_n^{(j,s_j)} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad A_n^{(jk)} = 0 \ (j \neq k). \tag{3.11}$$

Note that in case $\mu = 1$, the condition on $A_n^{(jk)}$ for $j \neq k$ becomes void!

Remark 5: On one hand, for theoretical arguments we may assume without loss in generality that the number n_0 introduced above is as large as we need it to be; however, for more computational purposes it is convenient to keep it as small as possible. So in the discussion to follow, we shall consider an arbitrarily fixed value n_0, but keep in mind that we might have arranged $n_0 \geq \nu$, because if not, then we may apply Theorem 9 to obtain a system with larger n_0, or one with superior leading term, *and the latter* *case can occur only finitely many times.* Hence after a finite number of steps we arrive at a system where, when normalizing it up to some power z^{-n_0}, it automatically satisfies (3.11). Also recall from Remark 4 (p. 47) that increasing n_0 can be done without changing the leading terms of the system. $\diamond$

To proceed, let (3.3) be reduced up to z^{-n_0}, and consider the finite set of rational numbers of the form p/q, $1 \le p \le q \le s_1$ (recall that by assumption s_1, the size of N_1, is at least as large as the sizes s_j of the other blocks N_j of A_0). For each such p/q, assume from now on that we have chosen p and q to be co-prime, i.e., having no nontrivial common divisor. The shearing transformation

$$
\begin{aligned}
T(z) &= \operatorname{diag}[T_1(z), \ldots, T_\mu(z)], \\
T_j(z) &= \operatorname{diag}[1, z^{p/q}, z^{2p/q}, \ldots, z^{(s_j-1)p/q}]
\end{aligned}
\tag{3.12}
$$

then produces a ramified system (except for $p = q = 1$), whose coefficient matrix may be denoted by $\hat{B}_{p/q}(z)$. The elements of this matrix in those positions corresponding to 1's in the matrix A_0 then are formal Laurent series in $z^{-1/q}$ beginning with the term $z^{r-p/q}$. Elements of $\hat{B}_{p/q}(z)$ in other positions may or may not involve higher rational powers of z – if they do, then *we discard the corresponding rational p/q as inadmissible.* This may eliminate many values of p/q, but we observe that *at least the smallest possible value $p/q = 1/s_1$ remains admissible.* If several admissible values remain, we take the largest one. For this admissible p/q, the matrix $\hat{B}_{p/q}(z)$ may be written in the form

$$
\hat{B}_{p/q}(z) = \hat{B}(z) = z^r \sum_{n=p}^{\infty} B_n z^{-n/q}, \tag{3.13}
$$

If $p/q = 1$ (hence $p = q = 1$), then the transformed equation is unramified and of Poincaré rank $r - 1$, so we consider this an improvement, and then apply to the resulting system the same arguments as above (depending upon its new leading term having several distinct eigenvalues or not). If this is not so, we show that the leading term B_p either is nilpotent or has several distinct eigenvalues:

Lemma 5 *Let a system* (3.3) *with coefficients as in* (3.11) *be given, let p/q be determined as above, and let the transformed equation be written as in* (3.13)*. Then we have*

$$
e^{-2n\pi i/q} B_n = D^{-1} B_n D, \quad n \ge p, \tag{3.14}
$$

with $D = T(e^{2\pi i})$ – note that this is not the identity matrix except for $q = 1$. In particular, the spectrum of the leading term B_p is closed with respect to multiplication with $e^{2\pi i/q}$, i.e. λ is an eigenvalue of B_p if and only if $\lambda e^{2\pi i/q}$ is one, too. Hence, if $p/q < 1$, then B_p either is nilpotent or has more than one eigenvalue.

Proof: It can be concluded from $\hat{B}(z) = T^{-1}(z)\,[\hat{A}(z)\,T(z) - zT'(z)]$ and $T(ze^{2\pi i}) = T(z)\,D$ that $\hat{B}(ze^{2\pi i}) = D^{-1}\hat{B}(z)\,D$, from which follows (3.14). This then implies $\det(B_p - xI) = e^{2\pi i\nu p/q}\det(B_p - e^{-2\pi i p/q}xI)$, showing that the spectrum of B_p is closed with respect to multiplication with $e^{-2\pi i p/q}$, and since p and q are assumed to be co-prime, the same follows for $e^{2\pi i/q}$. □

Hence, according to the above lemma, the only case where we did not make any progress is when B_p, for the maximal admissible value p/q, is nilpotent. This case, however, cannot occur when the value n_0 has been large enough: Imagine that $n_0 \geq \nu$, then for $n \leq \nu$ the coefficients A_n are diagonally blocked. In this case either 1 is admissible, or for the maximal admissible value $p/q < 1$ we have a leading term B_p which is a direct sum of matrices of the form $N_j + C_j$, with N_j as in (3.9), and C_j having zero rows except for possibly the first one. According to maximality of p/q, not all C_j can be the zero matrix, hence B_p cannot be nilpotent, as follows from Exercise 1.

We summarize the result of the preceding discussion as follows:

Theorem 10 *Given a formal system* (3.3) *with a nilpotent leading term, then one of the following two cases occurs:*

(a) *There exists a terminating meromorphic transformation $T(z)$ so that the transformed system has Poincaré rank smaller than r.*

(b) *There exist a $q \in \mathbb{N}$, not larger than the order of nilpotency of the leading term A_0, and a terminating q-meromorphic transformation $T(z)$, so that the transformed system has a non-nilpotent leading term whose spectrum is closed under multiplication by $e^{2\pi i/q}$.*

The minimal value for q and a transformation $T(z)$ can in both cases be found in an algorithmic manner following the steps described below.

What we have shown so far can be summarized in algorithmic form: Let a system (3.1), or more generally an arbitrary formal system (3.3), be given:

1. If the Poincaré rank of (3.3) is zero, or if the dimension ν equals one, *stop*. If not, find the number of distinct eigenvalues of the leading term A_0; if it turns out to be larger than one, *stop*. Otherwise, use a scalar exponential shift to go to a system with a nilpotent leading term and continue with step 2, taking the parameter $n_0 = 1$.

2. Normalize the system up to z^{-n_0}.

   (a) If (3.11) does not hold, use an unramified shearing transformation as in Proposition 6 (p. 47), resp. Theorem 9 to go to a system with superior leading term, and continue with step 2 and value $n_0 = 1$.

(b) If (3.11) holds, find the maximal admissible value p/q (≤ 1). For $p/q = 1$ use the shearing transformation (3.12) to go to a system of smaller Poincaré rank, and continue with step 1; otherwise, use the ramified shearing transformation (3.12) and a change of variable $z = w^q$ to go to a system whose leading term either has several eigenvalues (in which case we *accept* the transformed system and *stop*) or is nilpotent; in this case we *discard* the transformed system and return to the previous one, increase n_0 by one and continue with step 2.

The above algorithm terminates after finitely many steps. It produces finitely many transformations that we may combine into a single one denoted by $\tilde{T}(z)$, and a resulting system

$$w\tilde{x}' = \tilde{A}(w)\,\tilde{x}, \quad w = z^{1/q},\ q \in \mathbb{N}. \tag{3.15}$$

As can be read off from the results in this sections, the following two essentially different cases occur:

1. The system (3.15) has Poincaré rank $\tilde{r} = 0$; in this case all transformations used have been unramified (hence $q = 1$), and $\tilde{T}(z)$ has the form
$$\tilde{T}(z) = \mathrm{e}^{q(z)} T_1(z),$$
with a polynomial $q(z)$ of degree at most r and a terminating meromorphic transformation $T_1(z)$.

2. The system (3.15) has Poincaré rank $\tilde{r} \geq 1$; in this case $\tilde{r}$ and q are co-prime, the leading term of $\tilde{A}(w) = w^{\tilde{r}} \sum_0^\infty \tilde{A}_n\, w^{-n}$ has several eigenvalues, and $\tilde{T}(z)$ has the form
$$\tilde{T}(z) = \mathrm{e}^{q(z)} T_1(z)\, T(z),$$
with a polynomial $q(z)$ of degree at most r, a terminating meromorphic transformation $T_1(z)$, and a ramified shearing transformation $T(z)$ as in (3.12). Moreover, the coefficients $\tilde{A}_n$ satisfy
$$\mathrm{e}^{-2(n-\tilde{r})\pi i/q} \tilde{A}_n = T^{-1}(\mathrm{e}^{2\pi i})\, \tilde{A}_n\, T(\mathrm{e}^{2\pi i}), \quad n \geq 0.$$

Now, consider a convergent system (3.1) instead of a formal one. Then the system (3.15) will also converge. If the second one of the above two cases occurs, we may then apply the Splitting Lemma to (3.15). Doing so, we obtain a diagonally blocked system, whose blocks are formal of Gevrey order $1/\tilde{r}$. We shall show in the next section that this formal system can then be transformed into a convergent one, using a diagonally blocked formal analytic transformation of the same Gevrey order. Combining all the transformations used into a single one, we then have obtained what we shall define in the following section as a *highest-level formal fundamental solution.*

Exercises:

1. For a nilpotent Jordan block N and a constant matrix C with nonzero entries in the first row only, compute the characteristic polynomial of $A = N + C$ and show that A is nilpotent if and only if $C = 0$. Matrices of this form will also be called *companion matrices*, although they are slightly different from the ones named so before.

2. For the simplest nontrivial situation of $\nu = 2$, verify Theorem 10 (p. 50) and give explicit conditions under which each case occurs.

3. For a system (3.3) with a nilpotent leading term, give examples showing that the value for q in Theorem 10 (p. 50) can be any natural number strictly smaller than the order of nilpotency of A_0.

4. For a system (3.1) with nilpotent leading term, let $X(z)$ be an arbitrary fundamental solution of (3.1). For S as in Exercise 4 on p. 15, show existence of $c, a, \delta > 0$ so that

$$\|X(z)\| \le c\, e^{a|z|^{r-\delta}}, \qquad z \in S.$$

3.4 Transformation to Rational Form

Given a system (3.1), then in case (b) of Theorem 10 (p. 50) we can find a transformation which, after a change of variable, leads to a system of the same kind, but with a leading term having several eigenvalues. In this case, according to the Splitting Lemma, we can find a formal analytic transformation of Gevrey order s, with $1/s$ being the Poincaré rank of the new system, which splits this system into several smaller blocks. To continue, we wish to show that these formal systems may always be transformed into convergent ones, using formal analytic transformations of Gevrey order s – in fact, we shall show that the transformed system can be such that only finitely many of its coefficients are nonzero, hence *is a rational function* with poles (at most) at infinity and the origin.

Theorem 11 *Let* (3.3) *be a formal system of Gevrey order* $s = 1/r$. *Then for every sufficiently large* $N, M \in \mathbb{N}$, *a formal analytic transformation of Gevrey order* s *exists, which is of the form* $\hat{T}(z) = I + \sum_{n=N}^{\infty} T_n z^{-n}$, *so that the transformed system has a coefficient matrix of the form*

$$B(z) = z^r \sum_{n=0}^{N+M} B_n\, z^{-n}.$$

Proof: For the moment, assume that we had found $\hat{T}(z)$ and $B(z)$ as desired. Inserting formal expansions into (3.5) and equating coefficients then shows $B_n = A_n$, for $0 \le n \le N-1$, and

$$-(n-r)T_{n-r} = A_n - B_n + \sum_{m=N}^{n} (A_{n-m}T_m - T_m B_{n-m}), \tag{3.16}$$

for $n \ge N$, setting $T_n = 0$ for $n < N$. Using the notation

$$A(u) = \sum_{n=1}^{\infty} A_n \frac{u^{n-r}}{\Gamma(n/r)}, \quad T(u) = \sum_{n=N}^{\infty} T_n \frac{u^{n-r}}{\Gamma(n/r)},$$

we see that (3.4), (3.2) are equivalent to $A(u)$, $T(u)$ having positive radius of convergence. Slightly violating previous notation, we set $B(u) = \sum_{n=1}^{N+M} B_n u^{n-r}/\Gamma(n/r)$ and abbreviate

$$I(u,T,B) = \int_0^u \Big[A\Big((u^r - t^r)^{1/r}\Big)\, T(t) - T(t)\, B\Big((u^r - t^r)^{1/r}\Big)\Big]\, dt^r.$$

Then the integral equation

$$T(u)\, A_0 - (A_0 + ru^r I)\, T(u) = A(u) - B(u) + I(u,T,B) \tag{3.17}$$

can, by termwise integration and use of the Beta Integral (B.11) (p. 229), be seen to be equivalent to (3.16). So the proof of the theorem is equivalent to showing existence of $T(u)$ and $B(u)$, of the desired form, satisfying (3.17). For this, we use the space V of pairs (T,B) of matrix-valued functions that are holomorphic in $R(0,\rho)$, continuous up to the circle $|u| = \rho$, for $\rho > 0$ to be selected later, and have a zero resp. a pole at the origin of order at least $N-r$ resp. at most $r-1$. This is a Banach space with respect to the norm

$$\|(T,B)\| = \sup_{|u|\le\rho} (\|T(u)\|\, |u|^{r-N} + \|B(u)\|\, |u|^{r-1}).$$

Given *any* $(T,B) \in V$, we aim at finding a new pair $(\tilde{T},\tilde{B}) \in V$ so that

$$\tilde{T}(u)A_0 - (A_0 + ru^r I)\tilde{T}(u) = A(u) - \tilde{B}(u) + I(u,T,B). \tag{3.18}$$

Obviously, (3.18) is a system of linear equations for the matrix $\tilde{T}(u)$. However, its coefficient matrix has a determinant that is a polynomial in u vanishing at the origin; hence, the inverse matrix is rational and has a pole, say, of order p, at the origin, and for sufficiently small $\rho > 0$, the inverse matrix is holomorphic in $R(0,\rho)$ and continuous along the circular boundary. Hence the pair $(\tilde{T},\tilde{B})$ belongs to V, if the right-hand side of (3.18) vanishes at the origin of order at least $N+p-r$. By choosing $\tilde{B}(u)$ appropriately, we can arrange the right-hand side to vanish of order $N+M-r$ (and then $\bar{B}(u)$ is unique); hence we choose $M = p$. Thus, we have defined a mapping of V into itself, and a fixed point is a solution of (3.17). To apply Banach's fixed point theorem, we need the following estimates (for $|u| \le \rho$):

Taking any pair (T, B) with norm at most k, we have by definition of the norm in V that $\|T(u)\| \le k\,|u|^{N-r}$, $\|B(u)\| \le k\,|u|^{1-r}$ and, for suitable $a \ge 0$, $\|A(u)\| \le a\,|u|^{1-r}$. Integrating along the straight line segment from 0 to u and using the Beta Integral we obtain

$$\|I(u, T, B)\| \le c_N\,(a + k)\,k\,|u|^{N+1-r}, \qquad c_N = \frac{\Gamma(1/r)\Gamma(N/r)}{\Gamma((N+1)/r)}.$$

Putting $\tilde{B}(u) = \sum_{n=1}^{N} A_n u^{n-r}/\Gamma(n/r) + \tilde{B}_2(u)$, we find, using Cauchy's integral formula, that

$$\tilde{B}_2(u) = \frac{1}{2\pi i} \oint_{|t|=\rho} [A_2(t) + I(t, T, B)] k(u/t) \frac{dt}{t},$$

with $A_2(t) = \sum_{n=N+1}^{\infty} A_n u^{n-r}/\Gamma(n/r)$ and the kernel function $k(w) = w^{N+1-r}(1-w^M)/(1-w)$. Since $M = \sup_{|w|\le 1} |1-w^M|/|1-w|$, we can estimate the above integral to obtain $\|\tilde{B}_2(u)\| \le M[a_2 + c_N\,(a+k)\,k]|u|^{N+1-r}$, with a_2 depending only on $A_2(u)$. Hence the right-hand side of (3.18) is at most $(M+1)(a_2 + c_N\,(a+k)\,k)|u|^{N+1-r}$ (but note that by choice of $\tilde{B}_2(u)$, the right-hand side vanishes at the origin of order $N + M - r$). For u on the circle $|u| = \rho$, we have

$$|u|^{r-N}\|\tilde{T}(u)\| \le c(a_2 + c_N\,(a+k)\,k)$$

with a constant c that depends on ρ and M, but is independent of N. Due to the maximum principle, the same estimate then holds for smaller $|u|$. Combining the above estimates, one then obtains

$$\|(\tilde{T}, \tilde{B})\| \le b_1 + b_2 c_N (a + k) k,$$

with constants b_j that depend on $A(u)$, but are independent of N. Since $c_N \longrightarrow 0$ as $N \longrightarrow \infty$, one finds that for $k = 2b_1$ we may choose N so large that the norm of $(\tilde{T}, \tilde{B})$ is at most equal to k. In other words, the mapping $(T, B) \longmapsto (\tilde{T}, \tilde{B})$ maps the ball of V of radius k into itself. By similar estimates one can show that the mapping is contractive and hence has a unique fixed point. □

The problem of computing the transformation in the previous theorem is of a different nature than for those transformations in previous sections, since there it was, at least theoretically, possible to give the *exact values of any finite number of coefficients*, while here we use Banach's fixed point theorem, which is constructive, but gives *approximate values* only. However, since the earlier coefficients of our system remain unchanged, this will be of little influence on what we have in mind to do. Therefore, we may say that the previous theorem and the other results of this section *reduce the problem of finding a formal fundamental solution for a system* (3.1) *to the same problem for a finite number of systems of the same kind, but of strictly smaller dimension*. In the next section we briefly indicate a modification of the definition of the term *formal fundamental solution* that was first given in a different, but equivalent, formulation in [8] and which is very natural to consider, in particular in the discussion of Stokes' phenomena in sections to follow. Since Stokes' multipliers also can at best be approximated, it does not hurt in this context to have only approximate values for the coefficients of the system.

Exercises: In the following exercises, let a system (3.3) of Gevrey order $s = 1/r$ be given.

1. Assume that all coefficients A_n of (3.3) commute with one another. Prove that then Theorem 11 (p. 52) holds, with $N = 1$ and $M = r-1$ and a transformation $\hat{T}(z)$ commuting both with $\hat{A}(z)$ and $B(z)$. Note that this in particular settles the case of dimension $\nu = 1$.

2. Assuming that A_0 is diagonal, find the pole order p of the inverse matrix corresponding to the left hand side of (3.18).

3.5 Highest-Level Formal Solutions

The classical notion of regular versus irregular singularities distinguishes between cases where fundamental solutions have only a *moderate growth rate*, resp. grow exponentially. As shall become clear later on, the second case can naturally be divided into two subcases: Sometimes, the exponential growth is entirely due to the presence of *one* scalar exponential polynomial in a formal fundamental solution, while in other cases *several* such polynomials occur. Therefore, we shall use the following terminology:

1. We say that infinity is *an almost regular-singular point* of (3.1) if one can find a terminating meromorphic transformation $T(z)$ so that the transformed system has the form $B(z) = z^r \sum_{n=0}^{\infty} B_n z^{-n}$, with $B_j = \lambda_j I$ for $0 \le j \le r-1$. Note that then a scalar exponential shift transforms this system into one having a singularity of first kind at infinity.

2. We say that infinity is *an essentially irregular singularity* of (3.1) if it is not almost regular-singular.

3. A formal q-meromorphic transformation $\hat{F}(z)$, $q \in \mathbb{N}$, is called a *formal fundamental solution of highest level* for (3.1), if the following holds:

(a) The coefficient matrix $B(z) = \hat{F}^{-1}(z)[A(z)\hat{F}(z) - z\hat{F}'(z)]$ of the transformed system is of the form

$$B(z) = w^{qr} \sum_{n=0}^{n_0} B_n w^{-n}, \quad z = w^q,$$

for some $n_0 \in \mathbb{N}$, and the coefficients B_n are all diagonally blocked of some type $(s_1, \dots s_\mu)$, independent of n, with $\mu \geq 2$.

(b) For some integer p, $0 \leq p < rq$, which in case $q \geq 2$ is positive and co-prime with q, the coefficients $B_0, \dots, B_p$ have the form

$$\begin{aligned} B_j &= \lambda_j I \quad (0 \leq j \leq p-1), \\ B_p &= \operatorname{diag}[\lambda_1^{(p)} I_{s_1} + N_1, \dots, \lambda_\mu^{(p)} I_{s_\mu} + N_\mu], \end{aligned} \tag{3.19}$$

with distinct complex numbers $\lambda_\ell^{(p)}$ and nilpotent matrices N_ℓ.

(c) The transformation $\hat{F}(z)$ is of Gevrey order $s = q/(qr - p)$.

In the sequel, we shall abbreviate the term highest-level formal fundamental solution by HLFFS. The notion of HLFFS will turn out to be very important in what follows, so we wish to make the following comments:

- Having computed an HLFFS for a system (3.1), we have *(formally) partially decoupled the system* in the sense that we have reduced the problem of computing a fundamental solution of (3.1) to the same problem for several smaller systems. As we shall show later, the divergence of the transformation $\hat{F}(z)$ does not keep us from giving a clear analytic meaning to it, so that the process of decoupling will be not only a formal one.

- In the definition of HLFFS, the parameters $q, p, \lambda_0, \dots, \lambda_{p-1}$ and the pairs $(\lambda_j^{(p)}, s_j)$, $1 \leq j \leq \mu$, occurred. As a convenient way of referring to those quantities, we define

$$q_j(z) = \sum_{n=0}^{p-1} \lambda_n \frac{z^{r-n/q}}{r - n/q} + \lambda_j^{(p)} \frac{z^{r-p/q}}{r - p/q}, \quad 1 \leq j \leq \mu.$$

  These are polynomials in the qth root of z that shall play an important role later on, and the pairs $(q_j(z), s_j)$ will be referred to as *data pairs of the HLFFS.* We shall show in Chapter 8 that *any two HLFFS of the same system* (3.1) *have the same data pairs up to a renumeration*; in fact, more cannot hold in general, since given one HLFFS, we can block its columns into blocks of sizes s_k, and permuting these blocks can be seen to produce another HLFFS with the data pairs permuted correspondingly.

- We shall say that the data pairs of any HLFFS are *closed with respect to continuation*, provided that to every pair $(q_j(z), s_j)$, there is a (unique) pair $(q_{\tilde{j}}(z), s_{\tilde{j}})$ with $q_{\tilde{j}}(z) = q_j(z\mathrm{e}^{2\pi i})$ and $s_{\tilde{j}} = s_j$. Obviously this is equivalent to saying that $\lambda_j = 0$ whenever j is not a multiple of q, and that the pairs $(\lambda_j^{(p)}, s_j)$ are closed with respect to multiplication by $\exp[2\pi i/q]$, i.e., for suitable $\tilde{j}$ we have $(\lambda_{\tilde{j}}^{(p)}, s_{\tilde{j}}) = (\lambda_j^{(p)} \exp[2\pi i/q], s_j)$. We shall show in Chapter 8 that this always holds. Also note that, owing to the fact that p, q are co-prime when $p \neq 0$, we can recover p, q from the data pairs of an HLFFS.

Assuming that infinity is an essentially irregular singularity implies that in the algorithm described in Section 3.3 we end with case 1. This leads to the following result:

Theorem 12 *Every system* (3.1) *having an essentially irregular singularity at infinity possesses an HLFFS whose data pairs are closed with respect to continuation.*

Proof: From the algorithm formulated in Section 3.3, followed by an application of the Splitting Lemma together with Theorem 11 (p. 52), we conclude existence of a transformation $x = \mathrm{e}^{q(z)}\hat{T}(z)\,\tilde{x}$, with a polynomial $q(z)$ of degree at most r and a formal q-meromorphic transformation $\hat{T}(z)$ of Gevrey order $s = q/(qr-p)$, $p \geq 0$ and co-prime with q provided $p \neq 0$, such that the transformed system, after the change of variable $z = w^q$, has a coefficient matrix of the form $\tilde{B}(w) = w^{qr-p}\sum_0^{\tilde{n}_0} \tilde{B}_n\, z^{-n}$, with leading term $\tilde{B}_0$ having the form required for B_p. The scalar exponential shift $\exp[-q(w^q)]$ then leads to a system satisfying all the requirements for concluding that $\hat{T}(z)$ is an HLFFS, and its data pairs are closed with respect to continuation. □

The results of the previous sections enable us to compute an HLFFS in the following sense:

- First, follow the steps in the algorithm on p. 50 to compute a system whose leading term has several eigenvalues.
- Next, apply the Splitting Lemma to obtain several formal systems of Gevrey order $1/\tilde{r}$, with $\tilde{r}$ being their Poincaré rank.
- Finally, apply the iteration outlined in the proof of Theorem 11 (p. 52) to compute corresponding convergent systems; of these systems the first N coefficients will be known exactly, since they agree with corresponding coefficients of the formal systems, while the remaining nonzero coefficients can in principal be computed up to any degree of accuracy.

The HLFFS $\hat{F}(z)$ obtained by this algorithm is "known" in the sense that its first N terms are computed exactly, while finitely many other terms may be computed up to any desired degree of accuracy. Also, observe that the above algorithm computes an HLFFS for which the numbers λ_k in the definition vanish whenever k is not a multiple of q, so that the data pairs of the HLFFS are closed with respect to continuation. We shall show that the power series for $\hat{F}(z)$ in general has radius of convergence equal to zero, but is k-summable in *Ramis*' sense, for $k = 1/s$. To do so, we shall in the following chapters provide the necessary instruments from the *general theory of asymptotic power series*. In Chapter 8 we then will return to the investigation of HLFFS.

Exercises:

1. For $\nu = 1$, show that infinity always is an almost regular-singular point of (3.1).

2. Compute an HLFFS for (3.1), with $A(z) = zA_0 + A_1$ and

   (a) $A_0 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \lambda_1 \neq \lambda_2.$

   (b) $A_0 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \quad A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; b \neq 0.$

3. Assume that we are given a system (3.1) with A_0 having all distinct eigenvalues. Verify that the matrix $\hat{T}(z)$ in the formal fundamental solution obtained in Exercise 4 on p. 45 is an HLFFS.

4
Asymptotic Power Series

In this chapter we define the notion of *asymptotic expansions*, in particular *Gevrey asymptotics*, and we show their main properties. To do so, it will be notationally convenient to restrict ourselves to power series in z, hence statements on asymptotic behavior of functions are always made for the variable z tending to the origin in some region. The main applications will be to formal analytic or meromorphic transformations, i.e., series in $1/z$, but we trust that the reader will be able to make the necessary reformulation of the results.

The fact that formal analytic transformations have matrix coefficients does not cause any additional problems: If we develop a theory for scalar power series, it immediately carries over to series with matrix coefficients, since these are the same as a matrix whose entries are scalar power series. However, efforts have been made very recently to generalize the theory of multisummability to partial differential equations, or ordinary differential equations depending upon a parameter, and in this context one has to allow power series with coefficients in some space of functions in one or more variables. *Therefore, we shall consider a fixed but arbitrary Banach space $\mathbb{E}$ over $\mathbb{C}$, which we shall frequently assume to be a Banach algebra, and we will study functions, resp. power series, with values, resp. coefficients, in $\mathbb{E}$.* Many of the results to follow are true even for $\mathbb{E}$ being a more general space: Sometimes no topology on $\mathbb{E}$ is needed, so $\mathbb{E}$ might be an arbitrary vector space, or an algebra, over $\mathbb{C}$. Even when a topology is required, one may verify that instead of a norm one can make do with one or several seminorm(s) on $\mathbb{E}$, hence $\mathbb{E}$ may be a *Fréchet* space. We will, however, always assume that $\mathbb{E}$, equipped with some fixed norm $\|\cdot\|$, is a Banach space. The classical case then occurs for $\mathbb{E} = \mathbb{C}$, which is a Banach algebra under $\|\cdot\| = |\cdot|$, the modulus of complex numbers, and readers who are only interested in this case may always specialize the definitions and results by setting $\mathbb{E} = \mathbb{C}$, or $= \mathbb{C}^\nu$, or $= \mathbb{C}^{\nu\times\nu}$. For a brief description of the theory of holomorphic functions with values in a Banach space, and in particular of those results applied in this book, refer to Appendix B.

Aside from some of the books listed in Chapter 1, the theory of asymptotic power series has been presented by the following other authors: *Erdélyi* [101], *Ford* [103], *de Bruijn* [75], *Olver* [212], *Pittnauer* [219], *Dingle* [89], and *Sternin and Shatalov* [257].

Recently, several authors have developed a new theory under the name of *hyper-asymptotics*, which is not discussed here. As references, we mention [196, 209, 210].

4.1 Sectors and Sectorial Regions

We will deal with holomorphic functions, which in general have a branch point at the origin. Therefore, it is convenient to think of these functions as defined in sectorial regions on the Riemann surface of the natural logarithm, which is briefly described on p. 226 (Appendix B).

A *sector* on the Riemann surface of the logarithm will be a set of the form $S = S(d, \alpha, \rho) = \{z :\ 0 < |z| < \rho,\ |d - \arg z| < \alpha/2\}$, where d is an arbitrary real number, α is a positive real, and ρ either is a positive real number or ∞. We shall refer to d, resp. α, resp. ρ, as *the bisecting direction*, resp. *the opening*, resp. *the radius* of S. Observe that on the Riemann surface of the logarithm the value $\arg z$, for every $z \neq 0$, is uniquely defined; hence the bisecting direction of a sector is uniquely determined. In particular, if $\rho = \infty$, resp. $\rho < \infty$, we will speak of S having *infinite*, resp. *finite, radius*. It should be kept in mind that we do *not* consider sectors of infinite opening, nor an empty sector. If we write $S(d, \alpha, \rho)$, then it shall go without saying that d, α, ρ are as above. In case $\rho = \infty$, we mostly write $S(d, \alpha)$ instead of $S(d, \alpha, \infty)$. A *closed sector* is a set of the form

$$\bar{S} = \bar{S}(d, \alpha, \rho) = \{z :\ 0 < |z| \le \rho,\ |d - \arg z| \le \alpha/2\}$$

with d and α as before, but ρ a positive real number, i.e., never equal to ∞. *Hence closed sectors always are of finite radius, and they never contain the origin.* Therefore, closed sectors are not closed as subsets of $\mathbb{C}$, but are so as subsets of the Riemann surface of the logarithm, since this surface does not include the origin.

A region G, on the Riemann surface of the logarithm, will be named *a sectorial region*, if real numbers d and $\alpha > 0$ exist such that $G \subset S(d, \alpha)$, while for every β with $0 < \beta < \alpha$ one can find $\rho > 0$ for which $\bar{S}(d, \beta, \rho) \subset G$. We shall call α the opening and d the bisecting direction of G, and we frequently write $G(d,\alpha)$ for a sectorial region of bisecting direction d and opening α. Observe, however, that the notation $G(d,\alpha)$ *does not imply that the region is unbounded*, as it would be for sectors. Also, a sectorial region is not uniquely described by its opening and bisecting direction, but these two will be its essential characteristics. As an example, we mention that the open disc centered at some point $z_0 \neq 0$ and having radius equal to $|z_0|$ is a sectorial region of opening π and bisecting direction $\arg z_0$. For more examples, see the following exercises.

Exercises:

1. For $k > 0$ and $a \in \mathbb{C} \setminus \{0\}$, show that the mapping $z \mapsto a\,z^k$ maps sectorial regions of opening α and bisecting directions d to sectorial regions of opening $k\alpha$ and bisecting direction $kd + \arg a$.

2. For $k > 0$, $c \geq 0$ and $\tau \in \mathbb{R}$, show that the set of points described by the inequalities

$$k|\tau - \arg z| < \pi/2, \quad \cos(k[\tau - \arg z]) > c|z|^k, \tag{4.1}$$

   is a sectorial region of opening π/k and bisecting direction τ. In particular, picture the case of $c = 0$. Observe that the first inequality is needed to specify a sheet of the Riemann surface of the logarithm on which the sectorial region is situated.

3. For fixed $\alpha > 0$, and d in some open interval I, let sectorial regions $G(d,\alpha)$ be given. Show that their union is again a sectorial region, and find its opening and bisecting direction.

4.2 Functions in Sectorial Regions

Let G be a given sectorial region, and let $f \in \boldsymbol{H}(G,\mathbb{E})$ – hence $f(z)$ may be multi-valued if G has opening larger than 2π; see p. 227 for the definition of single- versus multi-valuedness. We say that *f is bounded at the origin*, if for every closed subsector $\bar{S}$ of G there exists a positive real constant c, depending on $\bar{S}$, such that $\|f(z)\| \leq c$ for $z \in \bar{S}$. We say that *f is continuous at the origin*, if an element of $\mathbb{E}$, denoted by $f(0)$, exists such that

$$f(z) \longrightarrow f(0), \qquad G \ni z \to 0,$$

with convergence being *uniform in every closed subsector*, meaning that for every $\bar{S} \subset G$ and every $\varepsilon > 0$ there exists $\rho > 0$ so that $\|f(z) - f(0)\| \leq \varepsilon$ for $z \in \bar{S}$ with $|z| < \rho$. Hence, continuity at the origin assures existence of a limit when approaching the origin along curves[1] staying within a closed subsector of G, while *no limit may exist for general curves in* G. Similarly, statements upon continuity, resp. a limit, as $z \to \infty$ are always meant to imply that convergence is uniform in closed subsectors.

We say that f, holomorphic in some sectorial region G, is *holomorphic at the origin*, if f can be holomorphically continued into a sector $S \supset G$ of opening more than 2π, and if f then is single-valued in S and bounded at the origin; the well-known result on removable singularities then implies that f has a convergent power series expansion about the origin; for the notion of essential singularities, poles, and removable singularities, compare Exercise 2 on p. 223.

We say that f is *differentiable at the origin*, if a complex number $f(0)$ exists so that the quotient $(f(z)-f(0))/z$ is continuous at the origin, in the sense defined above, and then its limit as $z \to 0$ in G is denoted by $f'(0)$ and named *the derivative of* f *at the origin*. In the exercises below, we shall show that differentiability of f at the origin is equivalent to continuity of f' there, and that $f'(0)$ is equal to the limit of $f'(z)$ as $z \to 0$ in G. Obviously, differentiability implies continuity of the function at the origin, but a function can be differentiable at the origin without being holomorphic there – for this, compare one of the exercises below. Moreover, we say that f is n-times differentiable at the origin, if its $(n-1)$st derivative $f^{(n-1)}(z)$ is differentiable at the origin. As follows from the exercises below, this is equivalent to $f^{(n)}(z)$ being continuous at the origin.

Let $S = S(d, \alpha)$ be a sector of infinite radius, and let $f(z)$ be holomorphic in S. Suppose that $k > 0$ exists for which the following holds true:

To every φ *with* $|d - \varphi| < \alpha/2$ *there exist* $\rho, c_1, c_2 > 0$ *such that for every* z *with* $|z| \geq \rho$, $|d - \arg z| \leq \varphi$,

$$\|f(z)\| \leq c_1 \exp[c_2 |z|^k].$$

Then we shall say that $f(z)$ *is of exponential growth at most* k (in S). This notion compares to that of (exponential) order as follows: If $f(z)$ is of exponential growth at most k (in S), then it either is of order less than k, or of order equal to k and of finite type, and vice versa (see the Appendix for the definition of order and type, and formulas relating both to the coefficients of an entire function). The set of all functions f, holomorphic and of exponential growth at most k in S and continuous at the origin, shall be denoted by $\boldsymbol{A}^{(k)}(S, \mathbb{E})$. As an example, we mention *Mittag-Leffler's function* $E_\alpha(z)$, defined on p. 233. It is an entire function of exponential order $k = 1/\alpha$ and finite type (equal to one); hence it is of exponential growth (at most) k in every sector of infinite radius. More generally, if $f_n \in \mathbb{E}$, $n \geq 0$, are such that for some $c > 0$ we have $\|f_n\| \leq c^n$, $n \geq 0$,

[1] Note that a *curve* is given by an arbitrary continuous mapping $x(t)$, $a \leq t \leq b$, while the term *path* means a *rectifiable* curve, i.e., a curve of finite length.

then

$$f(z) = \sum_{n=0}^{\infty} f_n\, z^n / \Gamma(1 + n/k)$$

is bounded by $E_{1/k}(c|z|)$, and therefore $f(z)$ is of exponential growth at most k in every sector of infinite radius. We shall occasionally say that a function f *is of exponential growth not more than* k *in a direction* d, if for some $\varepsilon > 0$ we have $f \in \boldsymbol{A}^{(k)}(S(d, \varepsilon), \mathbb{E})$.

Exercises: For some of the following exercises, let $\mathbb{E} = \mathbb{C}$ and define

$$g(z) = \int_0^{\infty} \mathrm{e}^{zt}\, t^{-t}\, dt = \int_0^{\infty} \exp[t(z - \log t)]\, dt, \tag{4.2}$$

integrating along the positive real axis. This interesting example is from *Newman* [203].

1. Let G be a sectorial region, and let $f(z)$ be analytic in G and so that $f'(z)$ is continuous at the origin. Let $f'(0)$ denote the limit of $f'(z)$ as $z \to 0$ in G. Show that $f(z)$ then is differentiable at the origin, and $f'(0)$ is equal to its derivative at the origin.

2. Let G be a sectorial region, and let $f(z)$ be analytic in G and differentiable at the origin. Let $f'(0)$ denote the derivative of f at the origin. Show that $f'(z)$ then is continuous at the origin, and $f'(0)$ is equal to the limit of $f'(z)$ as $z \to 0$ in G.

3. Let G be an arbitrary sectorial region, and consider $f(z) = z^{\mu} \log z$, for fixed $\mu \in \mathbb{C}$. For the largest integer k (if any) with $0 \le k < \mathrm{Re}\ \mu$, show that f is k-times, but not $(k+1)$-times, differentiable at the origin.

4. Show that $g(z)$ as in (4.2) is an entire function, and give the coefficients of its power series expansion in terms of some integral.

5. For $\mathrm{Im}\ z = \pi/2 + c$, $c > 0$, show that Cauchy's theorem allows in (4.2) to replace integration along the real axis by integration along the positive imaginary axis. Use this to show for these z that $|g(z)| \le 1/c$. Prove the same estimate for $\mathrm{Im}\ z = -(\pi/2 + c)$.

6. For every sector S of infinite radius, not containing the positive real axis, conclude that $g(z)$ is of exponential order zero in S.

7. Use Phragmen-Lindelöf's principle (p. 235) or a direct lower estimate of $g(x)$ for $x > 0$ to show that $g(z)$ cannot be of finite exponential growth in sectors S including the positive real axis; hence g is an entire function of infinite order.

8. For $c > 0$, let $f(z) = g(z+\pi/2+c)$. For every fixed $\phi \in \mathbb{R}$, show that $f(re^{i\phi})$ remains bounded as $r \to \infty$. Why does this not imply that f is of exponential order zero in every sector S?

4.3 Formal Power Series

Given a sequence $(f_n)_{n=0}^{\infty}$ of elements of the Banach space $\mathbb{E}$, the series $\hat{f}(z) = \sum_0^\infty f_n z^n$ is called a *formal power series* (in z), the term "formal" emphasizing that we do not restrict the coefficients f_n in any way; thus the radius of convergence of the series may well be equal to zero. The set of all such formal power series is denoted by $\mathbb{E}[[z]]$. We say that $\hat{f}$ *converges*, or *is convergent*, if $\rho > 0$ exists so that the power series converges for all z with $|z| < \rho$, defining a function $f(z)$, holomorphic in $D(0,\rho)$. We shall call $f(z)$ *the sum* of $\hat{f}$ whenever $\hat{f}$ converges, and we write $f = \mathcal{S}\hat{f}$. The set of all convergent power series will be denoted by $\mathbb{E}\{z\}$.

If $\hat{f}(z) = \sum f_n z^n$ is a formal power series so that for some positive c, K, and $s \geq 0$ we have

$$\|f_n\| \leq c\, K^n\, \Gamma(1+sn) \tag{4.3}$$

for every $n \geq 0$, then we say that $\hat{f}$ is a formal power series *of Gevrey order* s, and we write $\mathbb{E}[[z]]_s$ for the set of all such formal power series. Compare this to the definition of formal transformations of Gevrey order s on p. 39. Obviously, (4.3) holds for $s = 0$ if and only if the power series converges, hence $\mathbb{E}[[z]]_0 = \mathbb{E}\{z\}$. It follows from the exercises below that $\mathbb{E}[[z]]_s$, under natural operations, is a vector space closed under termwise differentiation, and a differential algebra in case of $\mathbb{E}$ being a Banach algebra.

Exercises: For formal power series in $\mathbb{E}[[z]]$ we consider the usual operations; also compare Section B.2 of the Appendix.

1. Show that $\mathbb{E}[[z]]$, with respect to addition and multiplication with scalars, is a vector space over $\mathbb{C}$.

2. In case $\mathbb{E}$ is a Banach algebra, show that $\mathbb{E}[[z]]$, with respect to multiplication of power series, is an algebra over $\mathbb{C}$, and is commutative if and only if $\mathbb{E}$ is so.

3. In case $\mathbb{E}$ is a Banach algebra, show that $\mathbb{E}[[z]]$, with respect to termwise derivation, is a differential algebra over $\mathbb{C}$; i.e., show that the map $\hat{f} \mapsto \hat{f}'$ is $\mathbb{C}$-linear and obeys the product rule $(\hat{f}\hat{g})' = \hat{f}'\hat{g} + \hat{f}\hat{g}'$.

4. Suppose that $\mathbb{E}$ is a Banach algebra with unit element e (hence $e = 1$ if $\mathbb{E} = \mathbb{C}$). Show that then $\hat{e}$, the power series whose coefficients are

all zero except for the constant term being equal to e, is the unit element in $\mathbb{E}[[z]]$.

5. Suppose that $\mathbb{E}$ is a Banach algebra with unit element e. Show that the invertible elements of $\mathbb{E}[[z]]$, i.e., those $\hat{f}$ to which $\hat{g}$ exists such that $\hat{f}\,\hat{g} = \hat{e}$, are exactly those whose constant term is invertible in $\mathbb{E}$ (i.e., is nonzero for $\mathbb{E} = \mathbb{C}$).

6. Assume that $\mathbb{E}$ is a Banach algebra. For arbitrary $s \geq 0$, show that $\mathbb{E}[[z]]_s$, with respect to the same operations as above, again is a differential algebra over $\mathbb{C}$.

7. Suppose that $\mathbb{E}$ is a Banach algebra with unit element e. For arbitrary $s \geq 0$, show $\hat{f} \in \mathbb{E}[[z]]_s$ invertible (in $\mathbb{E}[[z]]_s$) if and only if it is invertible in $\mathbb{E}[[z]]$, i.e., if and only if its constant term is invertible.

8. For arbitrary $s \geq 0$, show that for $\hat{f} = \sum f_n z^n \in \mathbb{E}[[z]]_s$ with $f_0 = 0$ we have $z^{-1}\hat{f}(z) = \sum_0^\infty f_{n+1} z^n \in \mathbb{E}[[z]]_s$.

4.4 Asymptotic Expansions

Given a function $f \in \boldsymbol{H}(G, \mathbb{E})$ and a formal power series $\hat{f}(z) = \sum f_n z^n \in \mathbb{E}[[z]]$, one says that $f(z)$ *asymptotically equals* $\hat{f}(z)$, as $z \to 0$ in G, or: *$\hat{f}(z)$ is the asymptotic expansion of $f(z)$* in G, if to every $N \in \mathbb{N}$ and every closed subsector $\bar{S}$ of G there exists $c = c(N, \bar{S}) > 0$ such that $|z|^{-N} \, \|f(z) - \sum_0^{N-1} f_n z^n\| \leq c$ for $z \in \bar{S}$. This is the same as saying that the *remainders*

$$r_f(z, N) = z^{-N}\Big(f(z) - \sum_{n=0}^{N-1} f_n \, z^n\Big)$$

are bounded at the origin, for every $N \geq 0$. If this is so, we write for short $f(z) \cong \hat{f}(z)$ in G, and whenever we do, it will go without saying that G is a sectorial region, $f \in \boldsymbol{H}(G, \mathbb{E})$ and $\hat{f} \in \mathbb{E}[[z]]$.

For the classical type of asymptotics with $\mathbb{E} = \mathbb{C}$ the results to follow are presented in standard texts as, e.g., [82, 281]. The proofs given there generalize to the Banach space situation, as we show now:

Proposition 7 *Given a sectorial region G, let f be holomorphic in G, with values in $\mathbb{E}$ and so that $f(z) \cong \hat{f}(z)$ in G for some $\hat{f}(z) = \sum f_n z^n \in E[[z]]$. Then the following holds true:*

(a) *The remainders $r_f(z, N)$ are all continuous at the origin, and*

$$r_f(z, N) \longrightarrow f_N, \quad G \ni z \to 0, \quad N \geq 0. \tag{4.4}$$

(b) *Suppose that the opening of G is larger than 2π and that $f(z)$ is single-valued. Then $f(z)$ is holomorphic at the origin, and $\hat{f}$ converges and coincides with the power series expansion of $f(z)$ at the origin.*

Proof: (a) Observe $z\, r_f(z, N+1) = r_f(z, N) - f_N$; hence $r_f(z, N+1)$ bounded at the origin implies (4.4).

(b) Under our assumptions, $f(z)$ is a single-valued holomorphic function in a punctured disc around the origin and remains bounded as $z \to 0$. Hence the origin is a removable singularity of f, i.e., $f(z)$ can be expanded into its power series about the origin. It follows right from the definition that the power series expansion is, at the same time, an asymptotic expansion, and from (a) we conclude that an asymptotic expansion is uniquely determined by $f(z)$. This proves $\hat{f}(z)$ to converge and be the power series expansion for $f(z)$. □

The following result shows that existence of an asymptotic expansion is equivalent to f's being infinitely often differentiable at the origin. However, observe that, unlike the case of holomorphic functions, existence of some derivative at the origin does not imply existence of higher derivatives.

Proposition 8 *Let f be holomorphic in a sectorial region G. Then the following statements are equivalent:*

(a) $f(z) \cong \hat{f}(z) = \sum_0^\infty f_n\, z^n$ *in G.*

(b) *The function f is infinitely often differentiable at the origin, and $f^{(n)}(0) = n!\, f_n$ for $n \ge 0$.*

(c) *All derivatives $f^{(n)}(z)$ are continuous at the origin, and*

$$f^{(n)}(z) \longrightarrow n!\, f_n, \quad G \ni z \to 0, \quad n \ge 0.$$

Proof: For the equivalence of (b) and (c), compare Exercises 1 and 2 on p. 63. Assuming (a), observe that according to Cauchy's integral formula for derivatives

$$\frac{n!}{2\pi i} \oint \frac{r_f(w, n+1)}{(1 - z/w)^{n+1}}\, dw = f^{(n)}(z) - n!\, f_n,$$

integrating along a circle around z. For z in a closed subsector $\bar{S}$ of G, the radius of this circle can be taken to be $\varepsilon|z|$, for some fixed ε, $0 < \varepsilon < 1$, depending on $\bar{S}$ but independent of z. Thus, $w/z = 1 + \varepsilon e^{i\phi}$ for some real number ϕ, so that the denominator of the integral can be bounded from below by some positive constant independent of z, but depending on $\bar{S}$. The standard type of estimate implies that the right-hand side tends to zero uniformly in $\bar{S}$; hence (c) follows. Conversely, let (c) hold. Then for $z, z_0 \in G$ and $N \geq 0$ we have

$$f(z) - \sum_0^{N-1} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = \frac{1}{(N-1)!} \int_{z_0}^{z} (z - w)^{N-1} f^{(N)}(w)\, dw.$$

Letting $z_0 \to 0$ and estimating the right-hand side of the resulting formula then implies (a). □

Let $\boldsymbol{A}(G, \mathbb{E}\,)$ be the set of all functions $f \in \boldsymbol{H}(G, \mathbb{E}\,)$, having an asymptotic expansion $\hat{f}(z)$. In view of the above proposition, part (a), to every $f(z) \in \boldsymbol{A}(G, \mathbb{E}\,)$ there is *precisely one* $\hat{f} \in \mathbb{E}\,[[z]]$ such that $f(z) \cong \hat{f}(z)$ in G. Therefore, we have a mapping

$$J: \ \boldsymbol{A}(G, \mathbb{E}\,) \longrightarrow \mathbb{E}\,[[z]], \quad f \longmapsto \hat{f} = J\, f, \tag{4.5}$$

mapping each $f(z)$ to its asymptotic expansion. The results to follow show that $\boldsymbol{A}(G, \mathbb{E}\,)$, under the natural operations and assuming $\mathbb{E}$ a Banach algebra, is a differential algebra, and J is a surjective homomorphism between $\boldsymbol{A}(G, \mathbb{E}\,)$ and $\mathbb{E}\,[[z]]$. However, examples given in the exercises below show that J is not injective – even if we consider regions of large opening. In the next section we are going to study another type of asymptotic expansions better suited to our purposes, since it will turn out that the corresponding map J, for regions of sufficiently large opening, is injective, however, not surjective.

Theorem 13 *Given a sectorial region G, suppose that $f_1, f_2 \in \boldsymbol{A}(G, \mathbb{E}\,)$. Then $f_1 + f_2 \in \boldsymbol{A}(G, \mathbb{E}\,)$ and $J(f_1 + f_2) = J\, f_1 + J\, f_2$. In other words,*

$$f_j(z) \cong \hat{f}_j(z) \text{ in } G, \quad 1 \leq j \leq 2,$$

implies

$$f_1(z) + f_2(z) \cong \hat{f}_1(z) + \hat{f}_2(z) \text{ in } G.$$

Proof: Follows directly from the definition. □

If $\mathbb{E}$ is a Banach algebra, then a product of any two elements of $\boldsymbol{A}(G, \mathbb{E}\,)$ again belongs to $\boldsymbol{A}(G, \mathbb{E}\,)$. This will be a corollary of the following more general result:

Theorem 14 *Suppose that $\mathbb{E}\,$, $\mathbb{F}$ are both Banach spaces and G is a sectorial region. Let $f \in \boldsymbol{A}(G, \mathbb{E}\,)$, $\alpha \in \boldsymbol{A}(G, \mathbb{C}\,)$, and $T \in \boldsymbol{A}(G, \mathcal{L}(\mathbb{E}\,, \mathbb{F}))$. Then*

$$\begin{aligned} T\, f &\in \boldsymbol{A}(G, \mathbb{F}), & J(T\, f) &= (J\, T)(J\, f), \\ \alpha\, f &\in \boldsymbol{A}(G, \mathbb{E}\,), & J(\alpha\, f) &= (J\, \alpha)(J\, f). \end{aligned}$$

Proof: We only prove the first one of the two statements; the second one can be shown following the same steps.

Let $(J\,T)(z) = \sum_0^\infty T_n\, z^n$, $(J\,f)(z) = \sum_0^\infty f_n\, z^n$. Given a closed subsector $\bar{S}$ of G and $N \in \mathbb{N}$, there exists $c_N > 0$ such that for every $z \in \bar{S}$ we have $\|r_f(z, N)\| \leq c_N$, $\|r_T(z, N)\| \leq c_N$. Then

$$\|r_{Tf}(z, N)\| \leq \|T(z)\|\, \|r_f(z, N)\| + \sum_{m=0}^{N-1} \|f_m\|\, \|r_T(z, N-m)\|.$$

Proposition 7 (p. 65) part (a) implies $\|f_m\| \leq c_m$ $(m \geq 0)$. Hence

$$\|r_{Tf}(z, N)\| \leq \sum_{m=0}^{N} c_m\, c_{N-m}, \tag{4.6}$$

completing the proof. □

If $\mathbb{E}$ is a Banach algebra, every $f \in \boldsymbol{A}(G, \mathbb{E})$ can be identified with an element of $\mathcal{L}(\mathbb{E}, \mathbb{E})$. Thus, the above theorem immediately implies:

Corollary to Theorem 14 *In case $\mathbb{E}$ is a Banach algebra, let $f_1, f_2 \in \boldsymbol{A}(G, \mathbb{E})$. Then $f_1\, f_2 \in \boldsymbol{A}(G, \mathbb{E})$ and $J(f_1\, f_2) = (J\, f_1)(J\, f_2)$. In other words,*

$$f_j(z) \cong \hat{f}_j(z) \text{ in } G, \quad 1 \leq j \leq 2,$$

implies

$$f_1(z)\, f_2(z) \cong \hat{f}_1(z)\, \hat{f}_2(z) \text{ in } G.$$

Note that $f(z) \cong \hat{f}(z)$ implies continuity of $f(z)$ at the origin, so that $\int_0^z \hat{f}(w)\, dw$ is well defined. For $\hat{f}(z) \in \mathbb{E}[[z]]$ we define $\int_0^z \hat{f}(w)\, dw$ by termwise integration.

Theorem 15 *Given a sectorial region G, suppose that $f(z) \cong \hat{f}(z)$ in G. Then*

$$f'(z) \cong \hat{f}'(z), \qquad \int_0^z f(w)\, dw \cong \int_0^z \hat{f}(w)\, dw \quad \text{in } G.$$

Proof: Follows easily from Proposition 8 (p. 66). □

The next result implies surjectivity of the mapping J defined in (4.5) and was shown for $\mathbb{E} = \mathbb{C}$ in [233], but the proof easily generalizes:

Theorem 16 (Ritt's Theorem) *Given any sectorial region G and any $\hat{f}(z) \in \mathbb{E}[[z]]$, there exists $f(z) \in \boldsymbol{A}(G, \mathbb{E})$ such that $f(z) \cong \hat{f}(z)$ in G.*

Proof: Without loss of generality, let G be a sector $S(d, \alpha)$ of infinite radius, and let $\hat{f}(z) = \sum f_n z^n$ be given. For $\beta = \pi/\alpha$, and $c_n = (\|f_n\|\, n!)^{-1}$

in case $f_n \neq 0$, resp. $c_n = 0$ otherwise, let $w_n(z) = 1 - \exp[-c_n/(z\mathrm{e}^{-id})^\beta]$. Using Exercise 1, we find $\|f_n\|\,|z|^n |w_n(z)| \le |z|^{n-\beta}/n!$ for every $n \ge 0$ and every $z \in G$. This proves absolute and locally uniform convergence of $f(z) = \sum_{n=0}^\infty f_n\, z^n\, w_n(z)$ in G, so that $f \in \boldsymbol{H}(G, \mathbb{E}\,)$. Moreover,

$$r_f(z,N) = f_N(z) - \sum_{n=0}^{N-1} f_n\, z^{n-N} \exp[-c_n/(z\mathrm{e}^{-id})^\beta],$$

where $f_N(z) = \sum_{n=N}^\infty f_n\, z^{n-N}\, w_n(z)$ is bounded in G, and the other terms tend to zero as $z \to 0$ in every closed subsector $\bar{S}$ of G, and hence are bounded at the origin. □

The following result characterizes the invertible elements in $\boldsymbol{A}(G, \mathbb{E}\,)$ in case $\mathbb{E}$ is a Banach algebra with unit element:

Theorem 17 *Let $\mathbb{E}$ be a Banach algebra with unit element e. Given a sectorial region G, suppose that $f(z) \cong \hat{f}(z)$ in G. Moreover, assume that the constant term f_0 of $\hat{f}(z)$ and all values $f(z)$, $z \in G$, are invertible elements of $\mathbb{E}$. Then*

$$f^{-1}(z) \cong \hat{f}^{-1}(z) \quad \text{in } G,$$

if $\hat{f}^{-1}(z)$ denotes the unique formal power series $\hat{g}(z)$ with $\hat{f}(z)\,\hat{g}(z) = \hat{e}$, the unit element of $\mathbb{E}\,[[z]]$.

Proof: Compare Exercise 5 on p. 65 for existence of $\hat{f}^{-1}(z) = \sum \tilde{f}_n\, z^n$. Given a closed subsector $\bar{S}$ of G, there exist $c_N > 0$ such that $\|r_f(z,N)\| \le c_N$ for $N \ge 0$ and $z \in \bar{S}$ and, using (4.4), $\|\tilde{f}_N\| \le c_N$ for every $N \ge 0$. The identity $r_{f^{-1}}(z,N)\, f(z) = -\sum_{m=0}^{N-1} \tilde{f}_m\, r_f(z, N-m)$, and the fact that for $z \in \bar{S}$ we have $\|f(z)\| \ge \delta > 0$, then imply

$$\|r_{f^{-1}}(z,N)\| \le \delta^{-1} \sum_{m=0}^{N-1} c_m c_{N-m},$$

completing the proof. □

Exercises:

1. Show $|1 - \mathrm{e}^{-z}| < |z|$ for z in the right half-plane.

2. Suppose $f(z)$ is holomorphic in a sectorial region G, and $f(z) \cong \hat{f}(z)$ in G for some $\hat{f} \in \mathbb{E}\,[[z]]$. If p is a natural number and $\tilde{G}$ is such that $z \in \tilde{G} \Longleftrightarrow z^p \in G$, show that $\tilde{f}(z) = f(z^p)$ is holomorphic in $\tilde{G}$, and $\tilde{f}(z) \cong \hat{f}(z^p)$ in $\tilde{G}$.

3. For arbitrary $b \in \mathbb{E}$, show $\mathrm{e}^{-z} b \cong \hat{0}$ in the right half-plane, where $\hat{0}$ stands for the power series with all coefficients equal to zero.

4. Suppose $f(z)$ is holomorphic in a sectorial region G, and $f(z) \cong \hat{0}$ in G. If $k > 0$ and $\tilde{G}$ is such that $z \in \tilde{G} \iff z^k \in G$, show that $\tilde{f}(z) = f(z^k)$ is holomorphic in $\tilde{G}$, and $\tilde{f}(z) \cong \hat{0}$ in $\tilde{G}$.

5. Show that the mapping J defined in (4.5) is never injective.

6. Let $f_n \in \boldsymbol{A}(G, \mathbb{E})$, $n \geq 0$. Assume that for every $m \geq 0$ the sequence $(f_n^{(m)})_n$ converges uniformly on every closed subsector of G. Show $f = \lim f_n \in \boldsymbol{A}(G, \mathbb{E})$.

4.5 Gevrey Asymptotics

Given $s > 0$, we say that a function f, holomorphic in a sectorial region G, *asymptotically equals* $\hat{f}(z) = \sum f_n z^n \in \mathbb{E}[[z]]$ *of Gevrey order* s, or $\hat{f}$ *is the asymptotic expansion of order* s *of* f, if to every closed subsector $\bar{S}$ of G there exist $c, K > 0$ such that for every non-negative integer N and every $z \in \bar{S}$

$$\|r_f(z, N)\| \leq c\, K^N\, \Gamma(1 + sN). \tag{4.7}$$

If this is so, we write for short $f(z) \cong_s \hat{f}(z)$ in G.

This terminology differs slightly from the one in [21] but agrees with the classical one used in most papers on Gevrey expansions. While for a general asymptotic, the bounds c_N for the remainders $\|r_f(z, N)\|$ may be completely arbitrary, we note that for a Gevrey asymptotic of order s their growth with respect to N is restricted. As we shall see, this has important consequences!

Observe that $f(z) \cong_s \hat{f}(z)$ in G implies $f(z) \cong \hat{f}(z)$ in G in the previous sense; hence from Proposition 7, part (a) (p. 65) we conclude that $f(z) \cong_s \hat{f}(z)$ in G implies $\hat{f}(z) \in \mathbb{E}[[z]]_s$. For $\tilde{s} > s$, Stirling's formula implies $\Gamma(1 + sN)/\Gamma(1 + \tilde{s}N) \to 0$ as $N \to \infty$; hence $f(z) \cong_s \hat{f}(z)$ in G implies $f(z) \cong_{\tilde{s}} \hat{f}(z)$ in G. Also note that (4.7) remains meaningful for $s = 0$ and is equivalent to $f(z)$ being holomorphic at the origin and $\hat{f}(z)$ being its power series expansion.

Proposition 9 *Let f be holomorphic in a sectorial region G, and let $s \geq 0$. Then the following statements are equivalent:*

(a) *$f(z) \cong_s \hat{f}(z)$ in G.*

(b) *All derivatives $f^{(n)}(z)$ are continuous at the origin, and for every closed subsector $\bar{S}$ of G there exist constants c, K such that*

$$\frac{1}{n!} \sup_{z \in \bar{S}} \|f^{(n)}(z)\| \le c\, K^n\, \Gamma(1 + sn)$$

for every $n \ge 0$.

Proof: Proceed analogously to the proof of Proposition 8 (p. 66). □

Let $\boldsymbol{A}_s(G, \mathbb{E})$ denote the set of all $f \in \boldsymbol{H}(G, \mathbb{E})$, having an asymptotic expansion of order s. The following results are the direct analogues to the ones in the previous section, proving that $\boldsymbol{A}_s(G, \mathbb{E})$ is again a differential algebra provided that $\mathbb{E}$ is a Banach algebra, and $J\colon \ \boldsymbol{A}_s(G, \mathbb{E}) \to \mathbb{E}\,[[z]]_s$ is a homomorphism.

Theorem 18 *Given a sectorial region G and $s \ge 0$, suppose that $f_1, f_2 \in \boldsymbol{A}_s(G, \mathbb{E})$. Then $f_1 + f_2 \in \boldsymbol{A}_s(G, \mathbb{E})$ and $J(f_1 + f_2) = J\, f_1 + J\, f_2$. In other words,*

$$f_j(z) \cong_s \hat{f}_j(z) \text{ in } G, \quad 1 \le j \le 2,$$

implies

$$f_1(z) + f_2(z) \cong_s \hat{f}_1(z) + \hat{f}_2(z) \text{ in } G.$$

Proof: Follows directly from the definition. □

Theorem 19 *Suppose that $\mathbb{E}$, $\mathbb{F}$ are both Banach spaces, G is a sectorial region, and $s \ge 0$. Let $f \in \boldsymbol{A}_s(G, \mathbb{E})$, $\alpha \in \boldsymbol{A}_s(G, \mathbb{C})$, and $T \in \boldsymbol{A}_s(G, \mathcal{L}(\mathbb{E}, \mathbb{F}))$. Then*

$$\begin{aligned} T\, f &\in \boldsymbol{A}_s(G, \mathbb{F}), & J(T\, f) &= (J\, T)(J\, f), \\ \alpha\, f &\in \boldsymbol{A}_s(G, \mathbb{E}), & J(\alpha\, f) &= (J\, \alpha)(J\, f). \end{aligned}$$

Proof: Set $c_N = c\, K^N\, \Gamma(1 + sN)$ in the proof of Theorem 14, and further estimate (4.6) by $c^2\, K^N\, \Gamma(1 + sN)\, (N + 1)$. Observing $1 + N \le 2^N$ then completes the proof. □

Corollary to Theorem 19 *In case $\mathbb{E}$ is a Banach algebra, let $f_1, f_2 \in \boldsymbol{A}_s(G, \mathbb{E})$. Then $f_1\, f_2 \in \boldsymbol{A}_s(G, \mathbb{E})$ and $J(f_1\, f_2) = (J\, f_1)(J\, f_2)$. In other words,*

$$f_j(z) \cong_s \hat{f}_j(z) \text{ in } G, \quad 1 \le j \le 2,$$

implies

$$f_1(z)\, f_2(z) \cong_s \hat{f}_1(z)\, \hat{f}_2(z) \text{ in } G.$$

Theorem 20 *Given $s > 0$ and a sectorial region G, suppose that $f(z) \cong_s \hat{f}(z)$ in G. Then*

$$f'(z) \cong_s \hat{f}'(z), \qquad \int_0^z f(w)\,dw \cong_s \int_0^z \hat{f}(w)\,dw \quad \text{in } G.$$

Proof: Follows directly from Proposition 9. □

Theorem 21 *Let $\mathbb{E}$ be a Banach algebra with unit element e. Given $s > 0$ and a sectorial region G, suppose that $f(z) \cong_s \hat{f}(z)$ in G. Moreover, assume that the constant term f_0 of $\hat{f}(z)$ and all values $f(z)$, $z \in G$, are invertible elements of $\mathbb{E}$. Then*

$$f^{-1}(z) \cong_s \hat{f}^{-1}(z) \quad \text{in } G,$$

if $\hat{f}^{-1}(z)$ denotes the unique formal power series $\hat{g}(z)$ with $\hat{f}(z)\,\hat{g}(z) = \hat{e}$, the unit element of $\mathbb{E}[[z]]$.

Proof: From Exercise 7 on p. 65 we conclude that $\hat{f}^{-1}(z) = \sum_0^\infty \tilde{f}_n\, z^n \in \mathbb{E}[[z]]_s$. Using the same arguments as in the proof of Theorem 17, one can obtain the necessary estimates. □

Exercises: Let $s > 0$, let $\hat{f} \in \mathbb{E}[[z]]$, and let $f(z)$ be holomorphic in a sectorial region G with values in $\mathbb{E}$.

1. (a) Given integers $p \geq 1$ and $q \geq 0$, assume that to every closed subsector $\bar{S}$ of G there exist $c, K > 0$ such that *for every $z \in \bar{S}$ and every N of the form $N = pM + q$, $M \in \mathbb{N}$,* we have (4.7). Show that then $f(z) \cong_s \hat{f}(z)$ in G.

   (b) Given $\varepsilon > 0$, assume that *to every closed subsector $\bar{S}$ of G with radius of $\bar{S}$ smaller than ε*, there exist $c, K > 0$ such that for every $z \in \bar{S}$ and every non-negative integer N we have (4.7). Show that then $f(z) \cong_s \hat{f}(z)$ in G.

2. Let p be a natural number. For $\tilde{G} = \{z|\ z^p \in G\}$, define $g(z) = f(z^p)$, $z \in \tilde{G}$, and $\hat{g}(z) = \hat{f}(z^p)$. Show that then $f(z) \cong_s \hat{f}(z)$ in G if and only if $g(z) \cong_{s/p} \hat{g}(z)$ in $\tilde{G}$.

3. Suppose $f(z) \cong_s \hat{f}(z)$ in G, and let the constant term of $\hat{f}(z)$ be zero. Show that then $z^{-1} f(z) \cong_s z^{-1} \hat{f}(z)$ in G.

4. Suppose $f(z) \cong_s \hat{f}(z)$ in $G(d, \alpha)$. Show that $f(z e^{\pm 2\pi i}) \cong_s \hat{f}(z)$ in a corresponding sectorial region $G(d \mp 2\pi, \alpha)$.

5. Given $s > 0$ and a *formal Laurent series* $\hat{f}(z) = \sum_{n=-m}^\infty f_n\, z^n$, for some $m \in \mathbb{N}$, show that the following two statements are equivalent:

(a) $f(z) - \sum_{n=-m}^{-1} f_n\, z^n \cong_s \sum_{n=0}^{\infty} f_n\, z^n$ in G,

(b) $z^m f(z) \cong_s \sum\limits_{n=0}^{\infty} f_{n-m}\, z^n$ in G.

Let either one serve as definition for $f(z) \cong_s \hat{f}(z)$ in G.

6. Assume that $\mathbb{E}$ is a Banach algebra. Let $\boldsymbol{A}_{s,m}(G, \mathbb{E})$ denote the set of functions $f(z)$ that are meromorphic in G, so that the number of poles of $f(z)$ in an arbitrary closed subsector of G is finite, and such that $f(z) \cong_s \hat{f}(z)$ for some formal Laurent series $\hat{f}$. Show that $\boldsymbol{A}_{s,m}(G, \mathbb{E})$ is a differential algebra. In case $\mathbb{E} = \mathbb{C}$ (a field), show that $\boldsymbol{A}_{s,m}(G, \mathbb{C})$ is a *differential field*, i.e., a differential algebra in which each element is invertible.

7. Show the existence of $f \in \boldsymbol{A}(G, \mathbb{E})$ that is not in $\boldsymbol{A}_s(G, \mathbb{E})$, for any $s \geq 0$.

8. Let $f_n \in \boldsymbol{A}_s(G, \mathbb{E})$, $n \geq 0$. Assume that for every $m \geq 0$ the sequence $(f_n^{(m)})_n$ converges uniformly on every closed subsector $\bar{S}$ of G. Moreover, assume existence of c, K, depending upon $\bar{S}$ but independent of n and m, so that $\|f_n^{(m)}(z)\| \leq c\, K^m\, m!\, \Gamma(1 + sm)$ for every $z \in \bar{S}$. Show $f = \lim f_n \in \boldsymbol{A}_s(G, \mathbb{E})$. Compare this to Exercise 6 on p. 70.

4.6 Gevrey Asymptotics in Narrow Regions

The following result is a version of Ritt's theorem, adapted to the theory of Gevrey asymptotics; its proof makes use of the so-called *finite Laplace operator*, which is defined and discussed in some detail in the following exercises.

Proposition 10 (Ritt's Theorem for Gevrey Asymptotics) *For $s > 0$, let $\hat{f}(z) \in \mathbb{C}[[z]]_s$ and a sectorial region G of opening at most $s\pi$ be arbitrarily given. Then there exists a function $f(z)$, holomorphic in G, so that $f(z) \cong_s \hat{f}(z)$ in G.*

Proof: Let $\hat{f}(z) = \sum_0^{\infty} f_n\, z^n$ and define $g(u) = \sum_0^{\infty} f_n\, u^n / \Gamma(1+sn)$, then $\hat{f}(z) \in \mathbb{C}[[z]]_s$ implies holomorphy of $g(u)$ for $|u|$ sufficiently small. Let d be the bisecting direction of G and define for $a = \rho e^{id}$, with sufficiently small $\rho > 0$, and $k = 1/s$:

$$f(z) = z^{-k} \int_0^a g(u) \exp[-(u/z)^k]\, du^k. \tag{4.8}$$

It follows from the exercises at the end of this section that this $f(z)$ has the desired properties. □

The above result ensures that the mapping $J:\ \boldsymbol{A}_s(G,\mathbb{E}\,) \to \mathbb{E}\,[[z]]_s$ is surjective if the opening of G is smaller than or equal to $s\pi$. It is, however, in this case not injective, as one learns from the following exercises.

Exercises: If not otherwise specified, let $g(u)$ be continuous for $\arg u = d$ and $0 \le |u| \le \rho$ with fixed d and $\rho > 0$, and define $f(z)$ by (4.8), for $k > 0$ and $a = \rho e^{id}$.

1. Show that $f(z)$ is holomorphic on the Riemann surface of the logarithm. In the literature, the mapping $g \longmapsto f$ is named *finite Laplace operator of order k*.

2. Assume that complex numbers g_n and real numbers $c_n \ge 0$ (for $n \ge 0$) exist so that for every $N \ge 0$ and every u as above $\|r_g(u,N)\| \le c_N$.

   (a) Setting $g(u) = 0$ for $\arg u = d$, $|u| > \rho$, show $\|r_g(u,N)\| \le \tilde{c}_N$, for every $N \ge 0$, every u with $\arg u = d$, and
   $$\tilde{c}_N = \max\{c_N, \sum_{n=0}^{N-1} c_n\, \rho^{n-N}\}.$$

   (b) For z with $\cos(k(d - \arg z)) \ge \varepsilon > 0$ and $N \ge 0$, set $f_n = g_n \Gamma(1 + n/k)$ and show
   $$\|r_f(z,N) z^n\| \le K_N = \varepsilon^{-1-N/k} \tilde{c}_N\, \Gamma(1 + N/k).$$

3. With c_n, K_n as above, assume for $s_1 \ge 0$ that $c_n \le c\, K^n \Gamma(1 + n s_1)$, $n \ge 0$, with sufficiently large $c, K \ge 0$, independent of n. Show that then $K_n \le \tilde{c} \tilde{K}^n \Gamma(1 + n s_2)$, with $s_2 = 1/k + s_1$ and sufficiently large $\tilde{c}, \tilde{K} \ge 0$ (independent of n, but depending on $\varepsilon > 0$). Use this to prove the above version of Ritt's theorem for Gevrey asymptotics.

4. For $k > 0$, $b \in \mathbb{E}$, and $c > 0$, let $f(z) = b \exp[-c z^{-k}]$. Show that $f(z) \cong_{1/k} \hat{0}$ in $S(0, \pi/k)$, with $\hat{0}$ being the zero power series.

5. Use the previous exercise to conclude that to every sectorial region G of opening not more than π/k there exists $f(z)$, holomorphic and nonzero in G, with $f(z) \cong_{1/k} \hat{0}$ in G.

6. Let G be a sectorial region *of arbitrary opening*, and let $f(z)$ be holomorphic in G with $f(z) \cong_{1/k} \hat{0}$ in G. To each closed subsector $\bar{S}$ of G, find $c_1, c_2 > 0$ so that $\|f(z)\| \le c_1 \exp[-c_2 |z|^{-k}]$ in $\bar{S}$.

7. Let G be a sectorial region of opening not more than $s\pi$, and let f be holomorphic in G with $f(z) \cong_s \hat{0}$ in G. Moreover, let $g(z)$ be holomorphic in G and so that for $k < 1/s$ and some real $c_1, c_2 > 0$ we have $\|g(z)\| \le c_1 \exp[c_2 |z|^{-k}]$ in G. Show $f(z) g(z) \cong_s \hat{0}$ in G.

4.7 Gevrey Asymptotics in Wide Regions

We have learned from Proposition 10 that the mapping $J: \ \boldsymbol{A}_s(G, \mathbb{E}) \to \mathbb{E}[[z]]_s$ is surjective for sectorial regions of opening at most $s\pi$. We now show that for wider regions J is injective – the fact that then it is no longer surjective will follow from results in the next chapter. The proposition we are going to show is due to *Watson* [282]:

Proposition 11 (WATSON'S LEMMA) *Suppose that G is a sectorial region of opening more than $s\pi$, $s > 0$, and let $f \in \boldsymbol{H}(G, \mathbb{E})$ satisfy $f(z) \cong_s \hat{0}$ in G. Then $f(z) \equiv 0$ in G.*

Proof: Let $\bar{S} = \bar{S}(d, \alpha, \rho)$ be any closed subsector of G of opening $\alpha > s\pi$. From Exercise 6 on p. 74 we conclude for suitably large constants c, K and $k = 1/s$

$$\|f(z)\| \le c\, \mathrm{e}^{-K|z|^{-k}}, \quad z \in \bar{S}.$$

In particular, $\|f(z)\|$ is bounded, say, by C, on $\bar{S}$. For $\kappa = \pi/\alpha\,(< k)$ and $z = z(w) = \mathrm{e}^{id}(\rho^{-\kappa} + w)^{-1/\kappa}$ we have $z \in \bar{S}$ for every w in the closed right half-plane. Thus, for arbitrary $x > 0$, the function $g(w) = \exp[x\, w] f(z(w))$ is bounded by C on the line $\mathrm{Re}\ w = 0$ and, because of $\kappa < k$, bounded by some, possibly larger, constant for $\mathrm{Re}\ w \ge 0$. Phragmén-Lindelöf's principle (p. 235) then implies

$$\|g(w)\| = \exp[x \ \mathrm{Re}\ w]\ \|f(z(w))\| \le C \quad \mathrm{Re}\ w \ge 0.$$

Letting $x \to \infty$ completes the proof. □

Given a sectorial region G, we shall say that a subspace $\boldsymbol{B}$ of $\boldsymbol{A}(G, \mathbb{E})$ is *an asymptotic space*, if the mapping $J: \ \boldsymbol{B} \to \mathbb{E}[[z]]_s$ is injective. In this terminology we may express the above proposition as saying that $\boldsymbol{A}_s(G, \mathbb{E})$ is an asymptotic space if the opening of G is larger than $s\pi$. This result, in a way, is the key to what will follow in the next chapters: Given a formal power series $\hat{f}(z)$ and a sectorial region G of opening larger than $s\pi$, if we succeed in finding a holomorphic function f with $f(z) \cong_s \hat{f}(z)$ in G, then *it is unique*, and it is justified, if not to say very natural, to consider this f as *a sum of some sort* for $\hat{f}$. While this abstract definition of a sum will be seen to have very natural properties, one certainly would like to have a way of somehow *calculating* f, and we shall see that this indeed can be done, at least in principle.

Exercises:

1. Let f be holomorphic for $|z| < \rho$, $\rho > 0$, and let $\hat{f}$ be its power series expansion. Conclude that then for every sectorial region G and every $s > 0$ we have $f(z) \cong_s \hat{f}(z)$ in G.

2. Show the following converse of the previous exercise: Let G be a sectorial region of opening more than $(2+s)\pi$, for some $s > 0$, and assume $f(z) \cong_s \hat{f}(z)$ in G. Show that then $\hat{f} \in \mathbb{C}\{z\}$.

5

Integral Operators

In this chapter we introduce a certain type of integral operators that shall play an important role later on. The simplest ones in this class are Laplace operators, while some of the others will be *Ecalle's acceleration operators*, that will be studied in detail in Chapter 11. All of them will be used in the next chapter to define some summability methods that, in the terminology common in this field, are called *moment methods*. As we shall prove, most of these methods are *equivalent* in the sense that they sum the same formal power series to the same holomorphic functions. The reason for investigating all these equivalent methods is that for particular formal power series it will be easier to check applicability of a particular method. Thus having all of them at our disposal gives a great deal of flexibility. Moreover, it also is of theoretical interest to know what properties of the methods are needed to sum a certain class of formal power series. It should, however, be noted that statements on methods being equivalent are here to be understood for summation of power series in *interior points* of certain regions, while the methods may be inequivalent when studying the same series at some boundary point.

Laplace and Borel operators are used in several different areas of mathematics, and are therefore defined and studied in many textbooks, such as [284] and others. They will also be the main tool in our theory of multisummability. Here, we only apply them to functions that are holomorphic in sectorial regions and continuous at the origin, which simplifies most of the proofs for their properties. For example, the identity theorem for holomorphic functions implies that formulas proven for some "small" set of complex numbers, e.g., an infinite set accumulating at some interior point of a region, immediately extend to the full region where all terms involved are holomorphic. Moreover, it will be convenient to slightly adjust the definition of Laplace operators, so that a power of the independent variable is mapped to the same power times a constant. For these reasons, we choose to include all the proofs for the properties of these operators.

A very recent paper dealing with Borel transform is by *Fruchard and Schäfke* [104]. For a discussion of related but more general integral operators see, e.g., *Braaksma* [65], *Schuitman* [246], or their joint article [73].

5.1 Laplace Operators

Let $S = S(d, \alpha)$ be a sector of infinite radius, and let $f \in \boldsymbol{A}^{(k)}(S, \mathbb{E})$, which we defined to mean that f is holomorphic and of exponential growth at most $k > 0$ in S and continuous at the origin. For τ with $|d - \tau| < \alpha/2$, the integral $\int_0^{\infty(\tau)} f(u) \exp[-(u/z)^k]\, du^k$, with integration along $\arg u = \tau$, converges absolutely and compactly in the open set $\cos(k[\tau - \arg z]) > c|z|^k$, if c is taken sufficiently large, depending upon f and τ. On the Riemann surface of the logarithm, the region described by this inequality in general has infinitely many connected components, one of which is specified by the inequalities (4.1) (p. 61). These describe a sectorial region $G = G(\tau, \pi/k)$ of opening π/k and bisecting direction τ, which has been discussed in Exercise 2 on p. 61. In G, the function

$$g(z) = z^{-k} \int_0^{\infty(\tau)} f(u) \exp[-(u/z)^k]\, du^k \tag{5.1}$$

is holomorphic. According to the definition of exponential growth, the constant c may be taken independent of τ, as long as τ is restricted to closed subintervals of $(d - \alpha/2, d + \alpha/2)$, but may go off to infinity when τ approaches the boundary values. Consequently, a change of τ essentially means a rotation of the region G, corresponding to holomorphic continuation of g. Therefore, $g(z)$ *is independent of* τ and holomorphic in the union of the corresponding regions $G(\tau, \pi/k)$, which is a, in general bounded, sectorial region $G(d, \beta)$, with $\beta = \alpha + \pi/k$. For short, we write $g = \mathcal{L}_k f$ and call $\mathcal{L}_k$ *the Laplace operator* of order k. We sometimes also say that $g = \mathcal{L}_k f$ is the *Laplace transform of order k of f*.

One immediately sees that $g(z)$ being the Laplace transform of order k of $f(u)$ is equivalent to $g(z^{1/k})$ being the Laplace transform of order 1 of $f(u^{1/k})$, in appropriate sectorial regions. Introducing a mapping s_α, for $\alpha > 0$, by $(s_\alpha f)(z) = f(z^\alpha)$, this property of Laplace operators can be conveniently stated as saying that, slightly more generally,

$$\mathcal{L}_{\alpha k} \circ s_\alpha = s_\alpha \circ \mathcal{L}_k, \tag{5.2}$$

for every $k > 0$.

The reason for the factor z^{-k} in front of the integral representation of $g(z)$ is that we want the Laplace transform of a power u^λ to be equal to $\Gamma(1+\lambda/k)\,z^\lambda$, as is to be shown in Exercise 1. Accordingly, *termwise application* of $\mathcal{L}_k$ to a formal power series $\hat f(u) = \sum_0^\infty f_n\,u^n$ produces the formal power series $\hat g(z) = \sum_0^\infty f_n\,\Gamma(1+n/k)\,z^n$, which we denote by $\hat{\mathcal{L}}_k \hat f$. The operator $\hat{\mathcal{L}}_k$ will frequently be called *the formal Laplace operator* of order k.

The following theorem shows that Laplace operators link very naturally with the notion of Gevrey asymptotics, making them the perfect tool for what will follow.

Theorem 22 *Let $f \in \boldsymbol{A}^{(k)}(S,\mathbb{E})$, for $k>0$ and a sector $S=S(d,\alpha)$, and let $g=\mathcal{L}_k f$ be its Laplace transform of order k, defined in a corresponding sectorial region $G = G(d,\alpha+\pi/k)$. For $s_1 \ge 0$, assume $f(z) \cong_{s_1} \hat f(z)$ in S, take $s_2 = 1/k + s_1$, and let $\hat g = \hat{\mathcal{L}}_k \hat f$. Then*

$$g(z) \cong_{s_2} \hat g(z) \quad \text{in } G.$$

Proof: For arbitrarily fixed $a=\rho e^{i\tau} \in S$, we may split the path of integration in (5.1) into the line segment from 0 to a and the ray from a to infinity, thus defining two functions $g_1(z)$, $g_2(z)$. From the exercises on p. 74, together with the ones below, we conclude $g_1(z) \cong_{s_2} \hat g(z)$, $g_2(z) \cong_{1/k} \hat 0$ in $G(\tau,\pi/k)$. This implies $g(z) \cong_{s_2} \hat g(z)$ in $G(\tau,\pi/k)$, owing to Theorem 18 (p. 71). Since G is the union of these regions as τ varies in the interval $(d-\alpha/2, d+\alpha/2)$, the proof is completed. □

Let $\boldsymbol{A}^{(k)}_{s_1}(S,\mathbb{E}) = \boldsymbol{A}^{(k)}(S,\mathbb{E}) \cap \boldsymbol{A}_{s_1}(S,\mathbb{E})$, i.e., the set of all holomorphic $\mathbb{E}$-valued functions on S that are of exponential growth not more than k and have a Gevrey asymptotic of order s_1 at the origin. The previous theorem then says that $\mathcal{L}_k$ maps $\boldsymbol{A}^{(k)}_{s_1}(S,\mathbb{E})$ into $\boldsymbol{A}_{s_2}(G,\mathbb{E})$, and $J \circ \mathcal{L}_k = \hat{\mathcal{L}}_k \circ J$. In the following sections we show that $\mathcal{L}_k$ is in fact bijective and its inverse equals *Borel's operator*.

Exercises:

1. Prove that the Laplace transform of u^λ, with complex λ, $\operatorname{Re}\lambda > 0$, equals $\Gamma(1+\lambda/k)\,z^\lambda$. Discuss for which other λ this statement holds as well.

2. For $f \in \boldsymbol{A}^{(k)}(S,\mathbb{E})$, $k>0$, show that $g=\mathcal{L}_k f$ is bounded at the origin.

3. For $f \in \boldsymbol{A}^{(k)}(S,\mathbb{E})$, $k>0$, take $a=\rho e^{i\tau} \in S$ and define

$$g(z) = z^{-k}\int_a^{\infty(\tau)} f(u)\exp[-(u/z)^k]\,du^k, \qquad z \in G(\tau,\pi/k).$$

Show $g(z) \cong_s \hat 0$ in $G(\tau,\pi/k)$ for $s=1/k$.

5.2 Borel Operators

Assume that $\tau \in \mathbb{R}$, $k, r > 0$, and $0 < \varepsilon < \pi$ are given. Let $\gamma_k(\tau)$ denote the path from the origin along $\arg z = \tau + (\varepsilon + \pi)/(2k)$ to some z_1 of modulus r, then along the circle $|z| = r$ to the ray $\arg z = \tau - (\varepsilon + \pi)/(2k)$, and back to the origin along this ray. In other words, the path $\gamma_k(\tau)$ is the boundary of a sector with bisecting direction τ, finite radius, and opening slightly larger than $\pi/(2k)$, and the orientation is negative. The dependence of the path on ε and r will be inessential and therefore is not displayed.

Let $G = G(d, \alpha)$ be a sectorial region of opening $\alpha > \pi/k$. For every τ with $|\tau - d| < (\alpha - \pi/k)/2$, we may choose ε and r so small that $\gamma_k(\tau)$ fits into the region G. If $f(z)$ is holomorphic in G and bounded at the origin, we define *the Borel transform of f of order k* by the integral

$$(\mathcal{B}_k f)(u) = \frac{1}{2\pi i} \int_{\gamma_k(\tau)} z^k f(z) \exp[(u/z)^k]\, dz^{-k}, \tag{5.3}$$

for $u \in S(\tau, \varepsilon/k)$. Observe that for these u the exponential function in the integral decreases along the two radial parts of the path, so that the integral converges absolutely and compactly and represents a holomorphic function of u. The integral operator $\mathcal{B}_k$ defined in this manner will be referred to as *the Borel operator of order k*.

From Cauchy's theorem we conclude that $\mathcal{B}_k f$ is independent of ε and r, and a change of τ results in holomorphic continuation of $\mathcal{B}_k f$. So we find that $\mathcal{B}_k f$ is also independent of τ, and therefore is holomorphic in the sector $S(d, \alpha - \pi/k)$. By a simple change of variable, one can show that, in analogy to (5.2),

$$s_\alpha \circ \mathcal{B}_k = \mathcal{B}_{\alpha k} \circ s_\alpha, \quad k, \alpha > 0.$$

An exercise below shows that termwise application of $\mathcal{B}_k$ to a formal power series $\hat{f}(z) = \sum_0^\infty f_n\, z^n$ produces the formal power series

$$(\hat{\mathcal{B}}_k \hat{f})(z) = \sum_0^\infty f_n\, z^n / \Gamma(1 + n/k),$$

and $\hat{\mathcal{B}}_k$ is called *the formal Borel operator.*

As for Laplace operators, we show that Borel operators also "respect" Gevrey asymptotics:

Theorem 23 *Let $G = G(d, \alpha)$ be an arbitrary sectorial region, let f be holomorphic in G, and for $s_1 \ge 0$ assume $f(z) \cong_{s_1} \hat{f}(z)$ in G. Let $k > 0$ be such that $\alpha > \pi/k$, so that $\mathcal{B}_k f$ is defined and holomorphic in $S = S(d, \alpha - \pi/k)$. Define s_2 by $s_2 = s_1 - k^{-1}$ if $1/s_1 < k$, resp. $s_2 = 0$ otherwise. Then*

$$(\mathcal{B}_k f)(u) \cong_{s_2} (\hat{\mathcal{B}}_k \hat{f})(u) \quad \text{in } S.$$

Proof: For τ close to d and $\gamma_k(\tau)$ in G, we have for sufficiently large $c, K > 0$ that $|r_f(z,N)| \le c\, K^N \Gamma(1+s_1 N)$, for every $N \ge 0$ and every $z \in \gamma_k(\tau)$. With $g = \mathcal{B}_k f$ we find that

$$u^N r_g(u,N) = \big(\mathcal{B}_k z^N r_f(z,N)\big)\,(u), \quad u \in S(\tau, \varepsilon/k).$$

Breaking $\gamma_k(\tau)$ into three pieces, the two radial parts and the circular arc of radius r, and estimating the resulting three integrals in the usual way, we find for $N \ge 0$ and arbitrarily fixed $u \in \bar{S}(\tau, \varepsilon/(2k), \rho)$, $\rho > 0$

$$|u^N r_g(u,N)| \le c\, K^N\, (2\pi)^{-1}\, \Gamma(1+s_1 N)\ (I_1 + I_2 + I_3),$$

with

$$I_1, I_3 \le k \int_0^r x^{N-1} \exp[-\hat{c}\,(|u|/x)^k]\, dx, \quad I_2 \le \tilde{c}\, r^N \exp[(|u|/r)^k],$$

for suitable constants $\hat{c}, \tilde{c} > 0$, independent of u, N and r. It suffices to consider large N; the final estimate then holds automatically for all N, with possibly enlarged constants. For sufficiently large N, however, we may choose $r = |u|\,(k/N)^{1/k}$, implying $I_2 \le \tilde{c}\,|u|^N\,(k/N)^{N/k}\, e^{N/k}$ and, after substituting $y = \hat{c}\,(|u|/x)^k$,

$$I_1, I_3 \le |u|^N\, \hat{c}^{N/k} \int_{\hat{c}N/k}^{\infty} y^{-1-N/k}\, e^{-y}\, dy \le |u|^N\, \hat{c}^{-1}\, (k/N)^{1+N/k}.$$

Using Stirling's formula, this shows, with suitably large $\tilde{C}, \tilde{K} > 0$, independent of u and N,

$$|r_g(u,N)| \le \tilde{C}\, \tilde{K}^N\, \Gamma(1+Ns_1)/\Gamma(1+N/k).$$

Another application of Stirling's formula and the fact that closed subsectors of S can be covered by finitely many sectors of the form $\bar{S}(\tau, \varepsilon/(2k), \rho)$ completes the proof. □

In case $1/s_1 \ge k$, Theorem 23 says that $\mathcal{B}_k f$ is holomorphic at the origin, and $\hat{\mathcal{B}}_k \hat{f}$ is its power series expansion. If $1/s_1 > k$, one can see, either from the final estimate in the proof or by looking at $\hat{\mathcal{B}}_k \hat{f}$ and estimating its coefficients, that $\mathcal{B}_k f$ is an entire function of exponential growth not larger than $(k^{-1} - s_1)^{-1}$. For a corresponding statement on the growth of $\mathcal{B}_k f$ in case $1/s_1 \le k$, see the next theorem in the following section.

Exercises:

1. For $f(z) = z^\lambda$, $\lambda \in \mathbb{C}$, show $(\mathcal{B}_k f)(u)$ exists and equals $u^\lambda/\Gamma(1+\lambda/k)$, for every $k > 0$.

2. Under the assumptions of Theorem 23, let G be a sector of infinite radius, and assume $f \in \boldsymbol{A}^{(\kappa_1)}(G, \mathbb{E})$, for some $\kappa_1 > 0$. Show that then $\mathcal{B}_k f \in \boldsymbol{A}^{(\kappa_2)}(S, \mathbb{E})$ for $1/\kappa_2 = 1/k + 1/\kappa_1$.

3. For $G(d, \alpha)$, f, k and τ as in (5.3), and sufficiently large $c > 0$, show that the curve β_y, parameterized by $z(t) = e^{i\tau}(c + it)^{-1/k}$, $-y \le t \le y$, is contained in G for arbitrary $y > 0$. Use Cauchy's integral theorem to conclude for $\arg u = \tau$ that

$$\lim_{y \to \infty} \frac{1}{2\pi i} \int_{\beta_y} z^k f(z) \exp[(u/z)^k] \, dz^{-k} = (\mathcal{B}_k f)(u). \tag{5.4}$$

5.3 Inversion Formulas

The following two theorems say that, roughly speaking, $\mathcal{B}_k$ is the inverse operator of $\mathcal{L}_k$, considered as mappings between certain spaces of analytic functions. In the first theorem we shall not worry about the exact region where $\mathcal{L}_k(\mathcal{B}_k f)(z)$ will be defined, compared to that of $f(z)$. Instead, we show $\mathcal{L}_k(\mathcal{B}_k f)(z) = f(z)$ for sufficiently many z, so that the Identity Theorem in Exercise 3 on p. 230 allows to conclude that they both are the same analytic function.

Theorem 24 *Let $G(d, \alpha)$ be a sectorial region, and let $k > \pi/\alpha$. For any $f \in \boldsymbol{H}(G, \mathbb{E})$ that is bounded at the origin, let*

$$g(u) = (\mathcal{B}_k f)(u), \quad u \in S = S(d, \alpha - \pi/k).$$

Then $g(u)$ is of exponential growth not more than k in S, so that $(\mathcal{L}_k g)(z)$ is holomorphic in a sectorial region $\tilde{G} = \tilde{G}(d, \tilde{\alpha})$, with $\pi/k < \tilde{\alpha}$, and

$$f(z) = (\mathcal{L}_k g)(z), \quad z \in \tilde{G} \cap G.$$

Proof: Split $\gamma_k(\tau)$ into three pieces, as in the proof of Theorem 23. Then the integrals over the radial parts uniformly tend to zero as $u \to \infty$ in $S(\tau, \varepsilon/(2k))$. In the integral over the circular arc, expand $\exp[(u/z)^k]$ into the exponential series and integrate termwise to see that this integral is an entire function of u of exponential growth not more than k. Altogether, this shows $\mathcal{B}_k f$ of exponential growth not more than k in $S(\tau, \varepsilon/(2k))$. Varying τ, we find the same in S.

To prove $f = \mathcal{L}_k g$, it suffices to do so for z with $\arg z = d$ and $|z|$ sufficiently small. For such z, we may insert the integral for $\mathcal{B}_k f$ into that for $\mathcal{L}_k g$, both with $\tau = d$. Interchanging the order of integration and then evaluating the inner integral gives

$$(\mathcal{L}_k \circ \mathcal{B}_k f)(z) = \frac{-k}{2\pi i} \int_{\gamma_k(d)} \frac{w^{k-1} f(w)}{w^k - z^k} dw.$$

The function $F_z(w) = w^{k-1} f(w)(w^k - z^k)^{-1}$ can be seen to have, in the interior of $\gamma_k(d)$, exactly one singularity, namely, at $w = z$, this being a pole of first order with residue $f(z)/k$. Since $\gamma_k(d)$ has negative orientation, the Residue Theorem in Exercise 3 on p. 224 completes the proof. □

Theorem 25 *For a sector $S = S(d, \alpha)$ of infinite radius and $k > 0$, let $f \in \boldsymbol{A}^{(k)}(S, \mathbb{E})$ and define $g(z) = (\mathcal{L}_k f)(z)$, $z \in G = G(d, \alpha + \pi/k)$. Then*

$$f(u) = (\mathcal{B}_k g)(u), \quad u \in S.$$

Proof: Using (5.4) for the representation of the Borel operator, fix y and insert (5.1), interchange the integrals and then evaluate the inner integral. Use the exercises below to complete the proof. □

Exercises:

1. Check that in Theorems 24 and 25 it suffices to prove the case $k = 1$ and $d = 0$; the general case then follows through suitable changes of variables.

2. For a sector $S = S(0, \alpha)$ of infinite radius, let $f \in \boldsymbol{A}^{(1)}(S, \mathbb{E})$. For sufficiently large $x_0 > 0$ and $t > 0$, show

$$\lim_{y \to \infty} \int_0^\infty f(u) \frac{e^{(t-u)(x_0+iy)} - e^{(t-u)(x_0-iy)}}{2\pi i\ (t-u)} du = f(t), \tag{5.5}$$

integrating along the positive real axis; observe that $u = t$ is a removable singularity of the integrand.

5.4 A Different Representation for Borel Operators

Here we give a different integral representation for Borel operators of order $k > 1/2$, using Mittag-Leffler's function as a kernel. To do so, we first prove the following well-known result upon the behavior of Mittag-Leffler's function as the variable $z \to \infty$; for a more complete asymptotic analysis of $E_s(z)$, compare the exercises below.

Lemma 6 (Asymptotic of Mittag-Leffler's function)
Let $0 < s < 2$. Then for $S_- = S(\pi, (2-s)\pi)$ we have

$$z\, E_s(z) \longrightarrow -1/\Gamma(1-s), \qquad S_- \ni z \to \infty, \tag{5.6}$$

while for $S_+ = S(0, s\pi)$ we have

$$E_s(z) = s^{-1} \exp[z^{1/s}] + \tilde{E}_s(z), \tag{5.7}$$

with $z\, \tilde{E}_s(z)$ bounded in S_+.

Proof: In the integral representation (B.19) (p. 233) one can deform the path of integration γ (owing to Cauchy's integral theorem) such that the two radial parts follow the two rays $\arg w = \pm(\pi+\varepsilon)/2$, for arbitrarily small $\varepsilon > 0$. Doing so, we find that for z with $|\pi - \arg z| \le (1 - s/2)\pi - s\varepsilon$ the denominator $w^s - z$ never vanishes, so that we may interchange integration and limit, by which we obtain (5.6), owing to (B.10) (p. 228). On the other hand, for $z \in S_+$ we can use the Residue Theorem to show

$$E_s(z) = s^{-1} \exp[z^{1/s}] + \tilde{E}_s(z), \quad |z| > 1,$$

where $\tilde{E}_s(z)$ has exactly the same integral representation (B.19), but for $|z| > 1$. From this representation, it is easy to see that $z\, \tilde{E}_s(z)$ remains bounded as $z \to \infty$ in S_+. □

Remark 6: Since $\exp[z^{1/s}]$ is bounded in the sectors $s\pi/2 \le \pm \arg z \le 3s\pi/2$, we observe that (5.7) remains valid in the sector $S(0, 3s\pi)$. This will be used in the proof of the next theorem. ◇

For s as above, consider any sectorial region $G = G(d, \alpha)$ of opening $\alpha > s\pi$, and let $f \in \boldsymbol{H}(G, \mathbb{E})$ be continuous at the origin. With $k = 1/s$ and $\gamma_k(\tau)$ as in Section 5.2, let

$$g(u) = \frac{-1}{2\pi i} \int_{\gamma_k(\tau)} E_s(u/z)\, f(z) \frac{dz}{z}. \tag{5.8}$$

According to Lemma 6, the integral is absolutely convergent for u with $|d - \arg u| < (\alpha - s\pi)/2$ and $\tau \approx \arg u$, and convergence is locally uniform there, so that $g(u)$ is analytic. We now show

Theorem 26 *For $0 < s < 2$ and any sectorial region $G = G(d, \alpha)$ of opening $\alpha > s\pi$, let $f \in \boldsymbol{H}(G, \mathbb{E})$ be continuous at the origin and define g by (5.8). Then we have*

$$g = \mathcal{B}_k f, \quad k = 1/s.$$

Proof: The integral for the Borel operator can be written as

$$(\mathcal{B}_k f)(u) = \frac{-1}{2\pi i} \int_{\gamma_k(\tau)} k f(z)\, \mathrm{e}^{(u/z)^k} \frac{dz}{z}.$$

Hence, according to (5.7), resp. Remark 6, we have

$$g(u) - (\mathcal{B}_k f)(u) = \frac{-1}{2\pi i} \int_{\gamma_k(\tau)} \tilde{E}_s(u/z) \frac{dz}{z}.$$

Since the integrand is bounded in the interior of the path of integration, we can use Cauchy's integral theorem to conclude $g - \mathcal{B}_k f = 0$. □

The above representation of Borel operators, aside from being interesting in its own right, also will serve as a starting point for more general integral operators in the next section. Concerning the restriction of s to the interval $0 < s < 2$, so that $k = 1/s > 1/2$, one should note that this is forced by the fact that Mittag-Leffler's function behaves differently for other values of s. In later chapters we shall see other results indicating that the value of $1/2$ is special in the sense that some aspects of the theory of summability of power series change when the parameter k is larger, resp. smaller, than $1/2$.

Exercises: Let $0 < s < 2$ and $S_- = S(\pi, (2-s)\pi)$.

1. For $N \geq 0$, show

$$E_s(z) = z^{1-N} E_s(z; N) - \sum_{n=1}^{N-1} z^{-n}/\Gamma(1-sn), \tag{5.9}$$

with $2\pi i\, E_s(z; N) = \int_\gamma \mathrm{e}^w\, w^{sN-1}\, (w^s - z)^{-1}\, dw$.

2. For $E_s(z; N)$ as above, show $z\, E_s(z; N) \to -1/\Gamma(1-sN)$, as $z \to \infty$ in S_-. Interpret this, together with (5.9), as an asymptotic for $E_s(1/z)$.

5.5 General Integral Operators

In this section, we are going to study some generalizations of Laplace resp. Borel operators. In detail, we are going to define pairs of integral operators, one being the inverse of the other on suitable function spaces, and each pair being associated with one particular "sequence of moments." To do so, we introduce pairs of functions serving as kernels for these operators.

KERNEL FUNCTIONS

A pair of $\mathbb{C}$-valued functions $e(z), E(z)$ will be called *kernel functions* if for some $k > 1/2$ the following holds:

- The function $e(z)$ is holomorphic in $S_+ = S(0, \pi/k)$, and $z^{-1}\, e(z)$ is *integrable at the origin*, meaning that the integral $\int_0^{x_0} x^{-1} |e(x e^{i\tau})|\, dx$ exists, for arbitrary $x_0 > 0$ and $2k|\tau| < \pi$. Moreover, for every $\varepsilon > 0$ there exist constants $c, K > 0$ such that
$$|e(z)| \le c \exp[-(|z|/K)^k], \qquad 2k|\arg z| \le \pi - \varepsilon. \quad (5.10)$$
- For positive real $z = x$, the values $e(x)$ are positive real.
- The function $E(z)$ is entire and and of exponential growth at most k. Moreover, in $S_- = S(\pi, \pi(2 - 1/k))$, i.e., the complement of $\overline{S_+}$ in $\mathbb{C}$, the function $z^{-1}\, E(1/z)$ is integrable at the origin in the above sense.
- The functions $e(z), E(z)$ are linked by the following *moment condition*: Define the *moment function* corresponding to the kernel $e(z)$ by
$$m(u) = \int_0^\infty x^{u-1}\, e(x)\, dx, \qquad \text{Re } u \ge 0. \qquad (5.11)$$
Note that the integral converges absolutely and locally uniformly for these u, so that $m(u)$ is holomorphic for Re $u > 0$ and continuous up to the imaginary axis, and the values $m(x)$ are positive real numbers for $x \ge 0$. Then the power series expansion of $E(z)$ equals
$$E(z) = \sum_0^\infty \frac{z^n}{m(n)}. \qquad (5.12)$$

The number k will sometimes be referred to as *the order* of the pair of kernel functions $e(z), E(z)$.

Observe that the kernel $E(z)$ is uniquely determined by $e(z)$ because of (5.12). Obviously, (5.10) implies $k\, m(n) \le c\, K^n\, \Gamma(n/k)$, $\quad n \ge 1$. On the other hand, by assumption we have that $E(z)$ is of exponential growth at most k, meaning by definition $|E(z)| \le \tilde c \exp[\tilde K |z|]^k$, for sufficiently large $\tilde c, \tilde K > 0$. From the proof of Theorem 69 (p. 233) we conclude that this implies existence of $\hat c, \hat K > 0$ so that $m(n) \ge \hat c\, \hat K^n \Gamma(1 + n/k)$, $n \ge 0$. Hence the moments $m(n)$ are *of order* $\Gamma(1+n/k)$ as n tends to infinity in the sense that $[m(n)/\Gamma(1 + n/k)]^{\pm 1/n}$ is bounded. In particular, this shows that the order of a pair of kernel functions is uniquely defined, and that the (entire) kernel $E(z)$ is exactly of exponential growth k, or in other words, is of order k and finite type. As a first example of such kernel functions we take $e(z) = k\, z^k \exp[-z^k]$; in this case $m(u) = \Gamma(1 + u/k)$ and $E(z) = E_{1/k}(z)$. Other examples of interest will follow in the exercises and the next section.

With the help of any such pair of kernel functions of order $k > 1/2$, we now define a pair of integral operators as follows:

Let $S = S(d,\alpha)$ be a sector of infinite radius, and let $f \in \boldsymbol{A}^{(k)}(S, \mathbb{E}\,)$, then for $|d - \tau| < \alpha/2$, the integral

$$(Tf)(z) = \int_0^{\infty(\tau)} e(u/z)\, f(u)\, \frac{du}{u} \tag{5.13}$$

converges absolutely and locally uniformly for z with (4.1) (p. 61), for sufficiently large $c > 0$. As for Laplace operators, a change of τ results in holomorphic continuation of Tf. Hence we conclude that Tf is holomorphic in a sectorial region $G(d, \alpha + \pi/k)$.

If $G = G(d,\alpha)$ is a sectorial region of opening larger than π/k, and $f \in \boldsymbol{H}(G, \mathbb{E}\,)$ is continuous at the origin, then we define, with $\gamma_k(\tau)$ as in the definition of Borel operators on p. 80,

$$(T^- f)(u) = \frac{-1}{2\pi i} \int_{\gamma_k(\tau)} E(u/z)\, f(z) \frac{dz}{z}. \tag{5.14}$$

We conclude as in the proof of Theorem 24 (p. 82) that $T^- f$ is holomorphic and of exponential growth at most k in $S(d, \alpha - \pi/k)$.

In case $e(z) = k\, z^k \exp[-z^k]$, the two integral operators coincide with Laplace resp. Borel operators. Even in general they have many properties in common with those classical operators, as we now show:

1. For $f(u) = u^\lambda$, with $\operatorname{Re} \lambda > 0$, hence $f(u)$ continuous at the origin, we have $Tf(z) = m(\lambda)\, z^\lambda$; to see this, make a change of variable in (5.13) and use (5.11).

2. For $f(u) = \sum_0^\infty f_n\, u^n$ being entire and of exponential growth at most k, the function Tf is holomorphic for $|z| < \rho$, with sufficiently small $\rho > 0$, and $(Tf)(z) = \sum_0^\infty f_n\, m(n)\, z^n$, $|z| < \rho$; to see this, check that termwise integration of the power series expansion of f is justified.

3. For $w \neq 0$ and $z \neq 0$ so that $|z/w|$ is sufficiently small, it follows from the above that

$$\frac{w}{w - z} = \int_0^{\infty(\tau)} e(u/z)\, E(u/w)\, \frac{du}{u}. \tag{5.15}$$

   This formula extends to values $w \neq 0$ and $z \neq 0$ for which both sides are defined. In particular, this is so for $\arg w \neq \arg z$ modulo 2π, since then we can choose τ so that $|\tau - \arg z| < \pi/(2k)$ and $|\pi - \tau + \arg w| < \pi(2 - 1/k)$, implying absolute convergence of the integral, according to the properties of kernel functions.

4. For a sectorial region $G = G(d,\alpha)$ of opening more than π/k, and $f \in \boldsymbol{H}(G, \mathbb{E}\,)$ continuous at the origin, the composition $h = T \circ T^- f$

is defined. Interchanging the order of integration and then evaluating the inner integral, using (5.15), implies

$$h(z) = \frac{-1}{2\pi i} \int_{\gamma_k(\tau)} \frac{f(w)}{w - z} \, dw = f(z),$$

since $\gamma_k(\tau)$ has negative orientation. Hence we conclude that T^- *is an injective integral operator, and T is its inverse.* Note, however, that we do not yet know that either operator is bijective; this will be shown in Theorem 30 (p. 92).

5. For Re $\lambda > 0$ and $f(z) = z^\lambda$, we conclude from (5.14) by a change of variable $u/z = w$, and using Cauchy's theorem to deform the path of integration:

$$(T^- f)(u) = u^\lambda \frac{1}{2\pi i} \int_\gamma E(w) \, w^{-\lambda-1} \, dw,$$

with γ as in Hankel's formula (p. 228). Hence $T^- f$ equals u^λ times a constant. Using that T is the inverse operator, we conclude that this constant equals $1/m(\lambda)$. In particular, this shows $m(u) \neq 0$ for Re $u > 0$. Moreover, we have the following integral representation for the reciprocal moment function:

$$\frac{1}{m(u)} = \frac{1}{2\pi i} \int_\gamma E(w) \, w^{-u-1} \, dw,$$

Compare this to Hankel's formula for the reciprocal Gamma function (B.10) (p. 228), and note that the integral also converges for u on the imaginary axis.

Exercises: In the following exercises, let $e(z), E(z)$ be a given pair of kernel functions of order $k > 1/2$, and let $\alpha > 0$.

1. For $e(z;\alpha) = z^\alpha \, e(z)$, find $E(z;\alpha)$ so that $e(z;\alpha), E(z;\alpha)$ are a pair of kernel functions of order k.

2. For $0 < \alpha < 2k$, define $e(z;\alpha) = e(z^{1/\alpha})/\alpha$, resp.

$$E(z;\alpha) = \frac{1}{2\pi i} \int_\gamma E(w) \frac{w^{\alpha-1}}{w^\alpha - z} \, dw. \tag{5.16}$$

Show that $e(z;\alpha)$ and $E(z;\alpha)$ are a pair of kernel functions of order k/α. Compare (5.16) to the integral representation of Mittag-Leffler's function (p. 233). Moreover, show that for the corresponding operators T and T_α one has $T \circ s_\alpha = s_\alpha \circ T_\alpha$, with s_α defined on p. 78.

3. Under the same assumptions as in the previous exercise, define

$$g(u) = \frac{-\alpha}{2\pi i} \int_{\gamma_k(\tau)} E((u/z)^\alpha; \alpha)\, f(z) \frac{dz}{z},$$

for $f \in \boldsymbol{H}(G, \mathbb{E})$ continuous at the origin, and G a sectorial region of opening larger than π/k. Show that $g = T^- f$.

5.6 Kernels of Small Order

In the previous section we restricted ourselves to kernels and corresponding operators of order $k > 1/2$. Here we generalize these notions to smaller orders.

Kernel Functions of Small Orders

A $\mathbb{C}$-valued function $e(z)$ will be called *a kernel function of order* $k > 0$ if we can find a pair of kernel functions $\tilde{e}(z), \tilde{E}(z)$ of order $\tilde{k} > 1/2$ so that

$$e(z) = \tilde{e}(z^{k/\tilde{k}})k/\tilde{k}, \quad z \in S(0, \pi/k). \tag{5.17}$$

From Exercise 2 on p. 88 we conclude that if a pair of kernels of *some* order $\tilde{k} > 1/2$ exists so that (5.17) holds, then there exists one for *any* such $\tilde{k}$. In particular, if k happens to be larger than $1/2$, then we can choose $\tilde{k} = k$, hence $e(z)$ is a kernel function in the earlier sense. Moreover, to verify that $e(z)$ is a kernel function of order k, we may always assume that $\tilde{k} = p\,k$, for a sufficiently large $p \in \mathbb{N}$. This then implies the following characterization of such kernel functions:

For arbitrary $k > 0$, $e(z)$ is a kernel function of order k, if and only if it has the following properties:

- The function $e(z)$ is holomorphic in $S_+ = S(0, \pi/k)$, and $z^{-1}\, e(z)$ is integrable at the origin. Moreover, for every $\varepsilon > 0$ there exist constants $c, K > 0$ such that (5.10) holds.
- For positive real $z = x$, the values $e(x)$ are positive real.
- For some $p \in \mathbb{N}$ with $p\,k > 1/2$, the function $E_p(z) = \sum_0^\infty z^n/m(n/p)$ is entire and of exponential growth not more than $p\,k$. Moreover, in the sector $S(\pi, \pi(2 - 1/(pk)))$ the function $z^{-1}\, E_p(1/z)$ is integrable at the origin.

Let a kernel function $e(z)$ of order k with $0 < k \le 1/2$ be given. Then we define the corresponding integral operator T as in (5.13). The definition of

T^-, however, cannot be given as in (5.14): While we can define an entire function $E(z)$ by means of (5.12), this function does not have the same properties as for $k > 1/2$. Therefore, we define the operator T^- as follows: Take any $\tilde{k} > 1/2$, then by definition, resp. Exercise 2 on p. 88, there exists a pair $\tilde{e}(z), \tilde{E}(z)$ of kernel functions of this order, such that (5.17) holds. Abbreviate $\alpha = \tilde{k}/k$ and set $\tilde{f} = s_\alpha f$, for $f \in \boldsymbol{A}^{(k)}(S, \mathbb{E})$ and s_α as on p. 78. Then $\tilde{f} \in \boldsymbol{A}^{(\tilde{k})}(\tilde{S}, \mathbb{E})$, for suitable $\tilde{S}$, and the functions $h = T f$ and $\tilde{h} = \tilde{T} \tilde{f}$ are related by $\tilde{h} = s_\alpha h$. In view of this relation between T and $\tilde{T}$, we now define:

- Let a kernel function $e(z)$ of order $0 < k \le 1/2$ be given. Choose $\tilde{k} > 1/2$ and let $\tilde{e}(z), \tilde{E}(z)$ and α be as above. For a sectorial region $G = G(d, \alpha)$ of opening larger than π/k, and any $f \in \boldsymbol{H}(G, \mathbb{E})$ that is continuous at the origin, we define with $\gamma_k(\tau)$ as in the definition of Borel operators on p. 80:

$$(T^- f)(u) = \frac{-1}{2\alpha\pi i} \int_{\gamma_k(\tau)} \tilde{E}((u/z)^{1/\alpha})\, f(z) \frac{dz}{z}.$$

For this definition to give sense, we have to show that the right-hand side does not depend upon the choice of $\tilde{k}$. This, however, follows directly from Exercise 3 on p. 89.

This definition of T^- has the following consequence: For $\tilde{f}(z) = s_\alpha f$ and $g = T^- f$, resp. $\tilde{g} = \tilde{T}^- \tilde{f}$ we obtain $\tilde{g}(u) = s_\alpha g$. This relation and the corresponding one for T which was shown above can be formally stated as

$$s_\alpha \circ T = \tilde{T} \circ s_\alpha, \quad s_\alpha \circ T^- = \tilde{T}^- \circ s_\alpha. \tag{5.18}$$

In particular, this shows that T maps a power u^n to $m(n)\, z^n$, while T^- is inverse to T, at least for such powers. This motivates the definition of formal operators $\hat{T}$, $\hat{T}^-$, acting on formal power series $\hat{f}$ by means of termwise application of T, resp. T^-. Consequently, $\hat{T}$ and $\hat{T}^-$ are inverse to one another.

Exercises: In the following exercises, let $e_j(z)$, $1 \le j \le 2$, be two kernel functions of the same order $k > 0$, and let $m_j(u)$ denote the corresponding moment functions.

1. For $k(z) = \sum_0^\infty m_1(n)\, z^n / m_2(n)$, show that the power series has positive, but finite, radius of convergence.

2. For $k > 1/2$ show that the function $k(z)$, defined above, can be holomorphically continued along all rays except for the positive real axis.

3. For $0 < k \le 1/2$, let $p \in \mathbb{N}$ be so that $p\,k > 1/2$. Define $k_p(z) = \sum_0^\infty m_1(n/p)\, z^n / m_2(n/p)$. Conclude from the previous exercise that $k_p(z)$ can be holomorphically continued along all rays except for the positive real axis. Use this fact to show the same for $k(z)$.

5.7 Properties of the Integral Operators

In this section, we consider fixed operators T, T^- of some order $k > 0$. From the definition of these operators in case of $k \le 1/2$, especially (5.18), one can see that in some of the proofs to follow it suffices to consider $k > 1/2$.

Using the corresponding formal operators defined above, we show that Theorems 22 and 23 in Sections 5.1 resp. 5.2 immediately generalize to the operators T, T^-:

Theorem 27 *Let $f \in \boldsymbol{A}^{(k)}(S, \mathbb{E})$, for $k > 0$ and a sector $S = S(d, \alpha)$, and let $g = Tf$ be given by (5.13), defined in a corresponding sectorial region $G = G(d, \alpha + \pi/k)$. For $s_1 \ge 0$, assume $f(z) \cong_{s_1} \hat{f}(z)$ in S, take $s_2 = 1/k + s_1$, and let $\hat{g} = \hat{T}\hat{f}$. Then*

$$g(z) \cong_{s_2} \hat{g}(z) \quad \textit{in } G.$$

Proof: By assumption, for every $\delta > 0$ there exist constants $c, K > 0$ such that $\|r_f(u,n)\| \le c\, K^n\, \Gamma(1+s_1 n)$ for every u with $|d - \arg u| \le \alpha - \delta$, $|u| \le 1$ and every $n \ge 0$. As was seen before, this implies $\|f_n\| \le c\, K^n\, \Gamma(1 + s_1 n)$, for every $n \ge 0$. Using that $f(u)$ is assumed to be of exponential growth not more than k, it is easily seen that (making c, K larger if needed)

$$\|r_f(u,n)\| \le c\, K^n\, \Gamma(1 + s_1 n) e^{K|u|^k},$$

for $|d - \arg u| \le \alpha - \delta$ and arbitrary $|u|$. Since the operator T maps $u^n\, r_f(u,n)$ to $z^n\, r_g(z,n)$, one can complete the proof estimating the integral in a standard manner. □

Theorem 28 *Let $G = G(d, \alpha)$ be an arbitrary sectorial region, let f be holomorphic in G, and for $s_1 > 0$ assume $f(z) \cong_{s_1} \hat{f}(z)$ in G. Let $k > 0$ be such that $\alpha > \pi/k$, so that $T^- f$ is defined and holomorphic in $S = S(d, \alpha - \pi/k)$. Define s_2 by $s_2 = s_1 - k^{-1}$ if $1/s_1 < k$, resp. $s_2 = 0$ otherwise. Then*

$$(T^- f)(u) \cong_{s_2} (\hat{T^-} \hat{f})(u) \quad \textit{in } \tilde{S}.$$

Proof: Observe that T^- maps $z^n\, r_f(z,n)$ to $u^n\, r_{T^- f}(u,n)$, and estimate as in the proof of Theorem 23 (p. 80). □

The first inversion theorem of Section 5.3 also generalizes to the situation of arbitrary integral operators T and T^-, as we show now.

Theorem 29 *Let $G(d, \alpha)$ be a sectorial region, and let $k > \pi/\alpha$. For $f \in \boldsymbol{H}(G, \mathbb{E})$ which is continuous at the origin, let*

$$g(u) = (T^- f)(u), \quad u \in S = S(d, \alpha - \pi/k).$$

Then $g(u)$ is of exponential growth not more than k in S, so that $(Tg)(z)$ is holomorphic in a sectorial region $\tilde{G} = \tilde{G}(d, \tilde{\alpha})$, with $\pi/k < \tilde{\alpha}$, and

$$f(z) = (Tg)(z), \quad z \in \tilde{G} \cap G.$$

Proof: For $k > 1/2$, the proof has been given in property 4 (p. 87). For smaller values of k, use the definition of T, T^- given in Section 5.6 in terms of operators $\tilde{T}$, $\tilde{T}^-$ of order $\tilde{k} > 1/2$, together with (5.18). □

To prove the analogue to the second inversion theorem of Section 5.3 is harder, owing to the fact that we have to use different rays of integration in the integral operator T in order to cover the path of integration $\gamma_k(\tau)$ used in the integral representation for T^-. Here, we give a proof that in the case of T's being a Laplace operator is considerably different from the proof of Theorem 25 (p. 83).

Theorem 30 *For a sector $S = S(d, \alpha)$ of infinite radius and $k > 0$, let $f \in \boldsymbol{A}^{(k)}(S, \mathbb{E})$ and define $g(z) = (Tf)(z)$, $z \in G = G(d, \alpha + \pi/k)$. Then $f = T^- g$ follows.*

Proof: Without loss of generality, assume that $d = 0$ and $k > 1/2$; the latter assumption can be made owing to (5.18). Let $\beta_\pm$ be the following path: from the origin along the ray $\arg z = \pm(\varepsilon + \pi)/(2k)$ to a point of modulus r, and then on the circle of radius r to the real axis, with fixed $\varepsilon, r > 0$. Then, for $0 < \delta < \alpha$ we may, for r and ε sufficiently small and t on the path $\beta_\pm$, represent $T f = g$ by the integral (5.13) with $\tau = \pm\delta$. Doing so, we define for positive real u

$$\begin{aligned} f_\pm(t) &= \frac{-1}{2\pi i} \int_{\beta_\pm} E(t/z) \int_0^{\infty(\pm\delta)} e(u/z)\, f(u)\, \frac{du}{u}\, \frac{dz}{z} \\ &= \int_0^{\infty(\pm\delta)} f(u)\, k_\pm(t, u) \frac{du}{u}, \\ k_\pm(t, u) &= \frac{-1}{2\pi i} \int_{\beta_\pm} E(t/z)\, e(u/z) \frac{dz}{z}. \end{aligned}$$

By definition of the paths $\beta_\pm$ we conclude that $f_+ - f_- = T^- g$. To evaluate the kernels $k_\pm(t, u)$, we make a change of variable $z = 1/w$, giving

$$k_\pm(t, u) = \frac{-1}{2\pi i} \int_{\tilde{\beta}_\pm} E(wt)\, e(wu) \frac{dw}{w},$$

with corresponding paths $\tilde{\beta}_\pm$ beginning at the point $w = 1/r$ and going off to infinity. The integrals remain the same when changing the paths of integration, provided that wu, for w on the path, stays within the sector S_+ where the kernel $e(wu)$ is defined, and wt, at least for large values of t, remains in S_- where then $E(wt)$ is bounded. By adding a corresponding integral from 0 to $1/r$, we can with help of Cauchy's theorem also integrate along the ray $\arg w = \mp(\pi+\varepsilon)/(2k)$. Doing so, and using (5.15), we obtain

$$k_\pm(t,u) + \frac{-1}{2\pi i}\int_0^{1/r} E(wt)\, e(wu)\frac{dw}{w} = \frac{1}{2\pi i}\frac{u}{t-u}. \tag{5.19}$$

This shows $k_+(t,u) = k_-(t,u)$, and $k_\pm(t,u)$ is analytic for $u \in S_+$ and arbitrary t, except for $t = u$. Hence we obtain

$$(T^- g)(t) = f_+(t) - f_-(t) = \int_\gamma f(u)\, k_\pm(t,u)\frac{du}{u},$$

where the path γ is from infinity to the origin along the ray $\arg u = -\delta$, and back to infinity along $\arg u = \delta$. For $|u| > |t|$, we can deform both rays to be along the real axis, and then these parts of the integral cancel. Therefore, we find that γ may as well be replaced by a closed Jordan curve encircling t and having negative orientation. This, together with (5.19) and Cauchy's formula, then implies $T^- g = f$. □

Exercises: In the following exercises, let $e(z)$ be a kernel function of order $k > 0$, and consider the corresponding integral operators T and T^-.

1. Let S be a sector of infinite radius, and let $\boldsymbol{A}$ denote a subspace of $\boldsymbol{A}^{(k)}(S, \mathbb{E})$ that is an asymptotic space, in the sense defined on p. 75. Show that then the image of $\boldsymbol{A}$ under the operator T is again an asymptotic space.

2. Let G be a sectorial region of opening more than π/k, and let $\boldsymbol{A}$ denote a subspace of $\boldsymbol{H}(G, \mathbb{E})$ which is an asymptotic space. Show that then the image of $\boldsymbol{A}$ under the operator T^- is again an asymptotic space.

5.8 Convolution of Kernels

In this section, we consider two pairs of operators T_j, T_j^- of order k_j with corresponding moment functions $m_j(u)$, $1 \le j \le 2$. We will try to find a third pair T, T^- of operators, so that T coincides with $T_1^+ \circ T_2^+$, resp. $T = T_2^+ \circ T_1^-$, at least when applied to the geometric series. Consequently, the corresponding $m(u)$ equals either the product $m_2(u)\, m_1(u)$ or the quotient $m_2(u)/m_1(u)$. In the first case the new operators clearly will have to have order $k = (1/k_1 + 1/k_2)^{-1}$, because it follows from Stirling's formula that $m(n)$ is of order $\Gamma(1 + n/k)$. In the second case, their order will be $k =$

$(1/k_2 - 1/k_1)^{-1}$, hence we here have to assume $k_1 > k_2$. Because of T^- being the inverse of T, at least for suitable function spaces, we only have to find T, resp. its kernel function $e(z)$.

The following lemma shows how one can recover $e(z)$ from its moment sequence $m(n)$. However, observe that we shall not be concerned with the harder question of how to characterize such $m(n)$ to which a kernel $e(z)$ exists.

Lemma 7 *Let a kernel function $e(z)$ of order k with corresponding operator T be given. For $f(u) = (1-u)^{-1}$, let $g = Tf$. Then $g(z)$ is holomorphic in for $-\pi/(2k) < \arg z < (2 + 1/(2k))\pi$, is asymptotic to $\hat{g}(z) = \sum_0^\infty m(n)\, z^n$ of Gevrey order $s = 1/k$ there, and $g(z) \to 0$ as $z \to \infty$. Moreover,*

$$g(z) - g(z e^{2\pi i}) = 2\pi i\, e(1/z), \quad |\arg z| < \pi/(2k). \tag{5.20}$$

Proof: Holomorphy of the function g and its behavior at the origin follow from Theorem 27 (p. 91), the behavior at infinity can be read off from the integral representation. To show (5.20), represent $g(z)$ resp. $g(ze^{2\pi i})$ by integrals (5.13) with $\tau = \varepsilon$ resp. $\tau = 2\pi - \varepsilon$, for (small) $\varepsilon > 0$, and use the Residue Theorem. □

Remark 7: Observe that, according to *Watson's Lemma* (p. 75), there is only one g with $g(z) \cong_{1/k} \sum m(n)\, z^n$ in a sector of opening so large as in the above lemma. So we indeed have shown that the kernel $e(z)$ is uniquely determined by (5.20). ◇

Theorem 31 *Let two kernel functions $e_j(z)$ of orders k_j, with corresponding moment functions $m_j(u)$ and operators T_j, $1 \le j \le 2$, be given. Then there is a unique kernel function $e(z)$ of order $k = (1/k_1 + 1/k_2)^{-1}$ with corresponding moment function $m(u) = m_1(u) m_2(u)$. For the corresponding integral operator T and $f(u) = (1-u)^{-1}$ we then have $T\, f = (T_1 \circ T_2)\, f = (T_2 \circ T_1)\, f$. In particular, the function $e(1/z)$ is given by applying T_1 to the function $e_2(1/u)$.*

Proof: Uniqueness follows from Lemma 7, resp. Remark 7. For existence, let $g(z) = (T_1 \circ T_2) f$, $f(u) = (1-u)^{-1}$, and define $e(z)$ by (5.20). Interchanging the order of integration in $(T_1 \circ T_2) f$, show that $e(1/z)$ is given by applying T_1 to $e_2(1/u)$. Then, check that $e(z)$ has the necessary properties for a kernel function of this order as listed on p. 89; in particular, note that the function $E_p(z)$ can be obtained by applying T_2^- to the corresponding function $E_{1,p}(z)$. □

As the final result in this context, we now find the kernel corresponding to the quotient of the moment functions:

Theorem 32 *Given two kernel functions $e_j(z)$ of order k_j with corresponding moment functions $m_j(u)$, $1 \le j \le 2$, assume $k_1 > k_2$. Then there exists a unique kernel function $e(z)$ of order $k = (1/k_2 - 1/k_1)^{-1}$ corresponding to the moment function $m(u) = m_2(u)/m_1(u)$. In particular, the function $e(1/u)$ is given by an application of T_1^- to the function $e_2(1/z)$.*

Proof: As above, uniqueness follows from Lemma 7, resp. Remark 7. For existence, let $f(u) = (1-u)^{-1}$ and define $g_2 = T_2 f$. Then $g = T_1^- g_2$ is holomorphic in $-\pi/(2k) < \arg w < (2 + 1/(2k))\pi$, and we define $e(z)$ by (5.20). Interchanging the order of integration, one obtains that $e(1/u)$ is given by an application of T_1^- to the function $e_2(1/z)$. To complete the proof, one can verify the necessary properties for $e(z)$, using Theorems 27 and 28. □

As an application of the last theorem, we mention that for $k_1 = \alpha > 1$, $e_1(z) = \alpha\, z^\alpha \exp[-z^\alpha]$, and $k_2 = 1$, $e_2(z) = z\, \mathrm{e}^z$, the function $C_\alpha(u) = u^{-1}\, e(u)$ equals the kernel of *Ecalle's* acceleration operator, which will be studied in detail in Chapter 11. More kernels are constructed in the following exercises.

Exercises:

1. Using the above results, verify existence of kernel functions corresponding to moment functions of the form

$$m(u) = \frac{\Gamma(1+s_1 u)\cdot \ldots \cdot \Gamma(1+s_\nu u)}{\Gamma(1+\sigma_1 u)\cdot \ldots \cdot \Gamma(1+\sigma_\mu u)}, \tag{5.21}$$

   with positive parameters s_j, σ_j satisfying $\sum_{j=1}^{\nu} s_j - \sum_{j=1}^{\mu} \sigma_j > 0$, and find the order of the kernels.

2. Using the above results, verify existence of kernel functions of order one corresponding to moment functions of the form

$$m(u) = \frac{\Gamma(\beta_1+u)\cdot \ldots \cdot \Gamma(\beta_\mu+u)\;\Gamma(1+u)}{\Gamma(\alpha_1+u)\cdot \ldots \cdot \Gamma(\alpha_\mu+u)}, \quad \mu \ge 1, \tag{5.22}$$

   with positive parameters α_j, β_j. Relate $E(z)$ to the generalized confluent hypergeometric function (p. 23).

3. Let $\hat f(z) = \sum_0^\infty m(n)\, z^n$, with $m(n)$ as in (5.21), resp. (5.22), and let k be the order of the corresponding kernel function. Show existence of a sectorial region G of opening larger than π/k and a function $f \in \boldsymbol{H}(G, \mathbb{C})$ with $f(z) \cong_{1/k} \hat f(z)$ in G.

4. For
$$m(u) = \frac{\Gamma(\alpha_1 + s_1 u) \cdot \ldots \cdot \Gamma(\alpha_\nu + s_\nu u)}{\Gamma(\beta_1 + \sigma_1 u) \cdot \ldots \cdot \Gamma(\beta_\mu + \sigma_\mu u)},$$
with positive parameters α_j, β_j, s_j, σ_j restricted by $\sum_{j=1}^{\nu} s_j = \sum_{j=1}^{\mu} \sigma_j$, determine the radius of convergence of the series
$$k(z) = \sum_{n=0}^{\infty} m(n)\, z^n .$$
Moreover, show that $k(z)$ admits holomorphic continuation along every ray other than the positive real axis.

6

Summable Power Series

In this chapter we shall present *Ramis's* concept of k-summability of formal power series [225, 226]. We shall also study the classical definition of *moment summability methods* in a form suitable for application to formal power series and relate these methods to k-summability. For some historical remarks, see Chapter 14.

A *general summability method* may be viewed as a linear functional $\mathbf{S}$ on some linear space X of sequences, or equivalently, series, with complex entries. In many cases the functional has the following representation: Let an infinite matrix $A = [a_{jk}]_{j,k=0}^{\infty}$ be given. Then we say that a series $\sum_0^\infty x_k$ is A-summable if the series $\sum_{k=0}^\infty a_{jk}\, x_k$ converge for every $j \geq 0$ and

$$\mathbf{S}_A\left(\sum x_k\right) = \lim_{j\to\infty} \sum_{k=0}^{\infty} a_{jk}\, x_k$$

exists. In this case the number $\mathbf{S}_A\left(\sum x_k\right)$ is called *the A-sum of the series* $\sum x_k$. Such summability methods are called *matrix methods*, and we say that the series *is summable by the method A, or for short, is A-summable.* The space X consisting of all A-summable series is called *the summability domain* of the method. Observe that this terminology gives good sense even in the more general situation of series $\sum x_k$ with x_k in a Banach space $\mathbb{E}$. One may even replace the entries a_{jk} of the matrix A by (continuous) linear operators on $\mathbb{E}$, but we shall not consider this here.

For our purposes it is more natural to replace the index j by a continuous parameter T. So instead of an infinite matrix A we have a sequence of functions $a_k(T)$, $T \geq 0$, which we again denote by A. Then the functional has the form $\mathbf{S}_A(\sum x_k) = \lim_{T\to\infty} \sum_{k=0}^{\infty} a_k(T)\, x_k$. If the series we want to sum is a formal power series, hence $x_k = f_k\, z^k$, then $\sum_{k=0}^{\infty} a_k(T)\, f_k\, z^k$, in case of convergence for, say, $|z| < \rho$, defines a family of holomorphic functions $a(z;T)$ on $D = D(0,\rho)$. It then is natural to restrict ourselves to cases where convergence for $T \to \infty$ is locally uniform in z, at least for z in some subregion G of D, so that we obtain a holomorphic function $f(z) = \lim_{T\to\infty} a(z;T)$ on G. In this case we say that the formal power series $\hat{f}(z)$ is *A-summable on the region* G, and we shall refer to the function $(\mathbf{S}_A \hat{f})(z) = f(z)$ as *the sum of* $\hat{f}(z)$, in the sense of the summation method A. Aside from the holomorphy of the sum, however, we need more properties of our summability method in order to make it suitable for differential equations: Suppose we know that the formal series $\hat{f}(z)$ satisfies some linear differential equation, and we have seen earlier that the radius of convergence of such a formal solution can be zero. If $\hat{f}(z)$ is indeed summable to $f(z)$ on G, we should like to conclude from general properties of the summation method that $f(z)$ also solves the differential equation. Moreover, the origin should be a boundary point of G, and we should like to conclude that $f(z)$ is asymptotic to $\hat{f}(z)$, perhaps of some Gevrey order $s > 0$. Thus, in order to make a summability method *suitable for formal solutions of ODE*, we require

- that its summability domain X is a differential algebra, containing all convergent power series,
- that the functional $\mathbf{S}$ is a homomorphism, and so is not only linear, but maps products to products and derivatives to derivatives, and
- that the map J, defined on p. 67, inverts $\mathbf{S}$.

For more details, see Ramis's discussion of such abstract summation processes in [229].

In particular, the requirement that products be summed to products is rarely satisfied for general summation methods, but we shall see that k-summability does have all the required properties, making it an ideal tool for treating formal solutions of ODE – *except that it is not powerful enough to sum all formal series arising as solutions of ODE!*

A trivial example of a summation method satisfying all the requirements listed above is as follows: Let $X = \mathbb{E}\{z\}$ and take $\mathbf{S} = \mathcal{S}$, mapping each convergent power series to its natural sum. Clearly, this method is too weak in that it applies to convergent series only. Constructing a summability method suitable for formal solutions of ODE may also be viewed as finding a way of extending $\mathcal{S}$ to larger differential algebras, since another natural requirement to make is that for convergent power series a summation method produces the natural sum.

6.1 Gevrey Asymptotics and Laplace Transform

In what follows we again consider a given Banach space $\mathbb{E}$. We have seen in Section 4.7 that the mapping $J: \boldsymbol{A}_s(G,\mathbb{E}) \to \mathbb{E}[[z]]_s$, owing to *Watson's Lemma*, is injective for sectorial regions G of opening more than $s\pi$, meaning that given a formal series $\hat{f}(z)$, there can be *at most one* function $f \in \boldsymbol{A}_s(G,\mathbb{E})$ with $f(z) \cong_s \hat{f}(z)$ in G. The following result will be used in the exercises below to show that in this case the mapping cannot be surjective, except for the trivial case of $\dim\mathbb{E} = 0$.

Theorem 33 *Let $s = 1/k > 0$ and $\hat{f} \in \mathbb{E}[[z]]_s$, hence $\hat{g}(u) = (\hat{\mathcal{B}}_k\hat{f})(u)$ converges for $|u|$ sufficiently small. Set $g = \mathcal{S}\,\hat{g}$; thus $g(u)$ is holomorphic in some neighborhood of the origin. Then, for every fixed real d the following two statements are equivalent:*

(a) *There exists a sectorial region $G = G(d,\alpha)$ with $\alpha > s\pi$, and $f \in \boldsymbol{A}_s(G,\mathbb{E})$ with $\hat{f} = J(f)$.*

(b) *There exists a sector $S = S(d,\varepsilon)$ for sufficiently small $\varepsilon > 0$ so that g admits holomorphic continuation into S and is of exponential growth not more than k there.*

Moreover, if either statement holds, then $f = \mathcal{L}_k g$ follows.

Proof: Follows immediately from Theorems 22–24 in the previous chapter. □

According to the theorem, existence of f as in (a) is linked to the holomorphic continuation of g plus its behavior when approaching infinity. In principle, (b) can be verified for a given formal series $\hat{f}$, and if satisfied, then the function f can be computed via Laplace transform.

Existence of power series that cannot be continued beyond their disc of convergence is well known; one such example is contained in the exercises below.

Exercises: Let $s > 0$ be given, and set $k = 1/s$.

1. For $\hat{g}(z) = \sum_0^\infty z^{2^n}$, show that $\hat{g}$ has radius of convergence equal to one. Moreover, show that $g = \mathcal{S}\,\hat{g}$ cannot be holomorphically continued beyond the unit disc.

2. For $\hat{g}$ as above, set $\hat{f} = \mathcal{L}_k\,\hat{g}$. Use Theorem 33 and the previous exercise to conclude that no $f \in \boldsymbol{A}_s(G,\mathbb{C})$ with $\hat{f} = J(f)$ can exist if the opening of G exceeds $s\pi$. Hence the map $J: \boldsymbol{A}_s(G,\mathbb{E}) \to \mathbb{E}[[z]]_s$, for $\mathbb{E} = \mathbb{C}$, is not surjective. Generalize this to arbitrary $\mathbb{E}$ with $\dim\mathbb{E} > 0$.

6.2 Summability in a Direction

Let $k > 0$, $d \in \mathbb{R}$ and $\hat{f} \in \mathbb{E}[[z]]$ be given. We say that $\hat{f}$ is *k-summable in direction d*, if a sectorial region $G = G(d, \alpha)$ of opening $\alpha > \pi/k$ and a function $f \in \boldsymbol{A}_{1/k}(G, \mathbb{E})$ exist with $J(f) = \hat{f}$. If this is so, then $\hat{f} \in \mathbb{E}[[z]]_{1/k}$ follows from results in Section 4.5, and *Watson's Lemma* (p. 75) guarantees uniqueness of $f \in \boldsymbol{A}_{1/k}(G, \mathbb{E})$. In view of Theorem 33, $\hat{f}$ is k-summable in direction d if and only if $\hat{g} = \hat{\mathcal{B}}_k \hat{f}$ converges and $g = \mathcal{S}(\hat{g})$ is holomorphic and of exponential growth at most k in a sector $\tilde{S} = S(d, \varepsilon)$, for some $\varepsilon > 0$. If this is so, we call the function $f = \mathcal{L}_k g$, integrating along rays close to d, the *k-sum of $\hat{f}$ in direction d* and write $f = \mathcal{S}_{k,d}\, \hat{f} = \mathcal{L}_k \circ \mathcal{S} \circ \hat{\mathcal{B}}_k\, \hat{f}$.

Remark 8: If we are given a series $\hat{f} \in \mathbb{E}[[z]]_{1/k}$ whose k-summability in direction d is to be investigated, and if we then want to compute its sum, we are presented with the following problems:

1. The function $g(z)$, locally given by the convergent series $\hat{\mathcal{B}}_k \hat{f}$, has to be holomorphically continued into a sector $S(d, \varepsilon)$, for some $\varepsilon > 0$, which typically will be small.

2. In this sector, we have to show that g is of exponential growth not larger than k.

3. We have to compute the integral $\mathcal{L}_k g$.

At first glance, one might think that item 1 might be the major problem. However, there are explicit methods for performing holomorphic continuation, once we know that g *is* holomorphic in $S(d, \varepsilon)$. Moreover, we shall show in Lemma 8 that we may even replace k by $\tilde{k} < k$, with $k - \tilde{k}$ sufficiently small; then the sum of $\hat{\mathcal{B}}_{\tilde{k}} \hat{f}$ *will be an entire function*, so that item 1 is no problem at all. This entire function, in general, will have too large an exponential growth in all directions but $\arg u = d$. So in a way, the main difficulty lies in the problem of verifying item 2. $\diamond$

As we shall show, this summation method has all the properties listed at the beginning of the chapter, plus several additional ones, and we begin with proving some which are direct consequences of the definition:

Lemma 8 *For every fixed $k > 0$ the following holds:*

(a) *Let $\hat{f}$ be convergent. Then for every d, the series $\hat{f}$ is k-summable in direction d, and $(\mathcal{S}_{k,d}\, \hat{f})(z) = (\mathcal{S}\, \hat{f})(z)$ for every z where both sides are defined.*

(b) *Let $\hat{f}$ be k-summable in direction d, and let $\varepsilon > 0$ be sufficiently small. Then $\hat{f}$ is k-summable in all directions $\tilde{d}$ with $|\tilde{d} - d| < \varepsilon$, and $(\mathcal{S}_{k,\tilde{d}}\, \hat{f})(z) = (\mathcal{S}_{k,d}\, \hat{f})(z)$ for every z where both sides are defined.*

(c) *Let $\hat{f}$ be k-summable in direction d, and let $\varepsilon > 0$ be sufficiently small. Then $\hat{f}$ is $(k-\varepsilon)$-summable in direction d, and $(\mathcal{S}_{k-\varepsilon,d}\,\hat{f})(z) = (\mathcal{S}_{k,d}\,\hat{f})(z)$ for every z where both sides are defined.*

Proof: For the first statement, use $f(z) = (\mathcal{S}\,\hat{f})(z) \cong_{1/k} \hat{f}(z)$ in S, for every $k > 0$ and every sector S of sufficiently small radius. To prove the second, use the definition and observe that for a sectorial region $G(d,\alpha)$ of opening $\alpha > \pi/k$ and $\varepsilon < \alpha - \pi/k$, there exists $G(\tilde{d},\alpha_\varepsilon) \subset G(d,\alpha)$ with $\tilde{d}$ as above and $\alpha_\varepsilon > \pi/k$. Finally, for (c) use that $\alpha > \pi/k$ implies $\alpha > \pi/(k-\varepsilon)$ for small $\varepsilon > 0$, and $\Gamma(1+N/k)/\Gamma(1+N/(k-\varepsilon)) \to 0$ as $N \to \infty$. □

The following lemma shall prove useful in later chapters where we will study multisummablility.

Lemma 9 *Let $\hat{f}$ be k_1-summable in direction d, let $k > k_1$, and define k_2 by $1/k_2 = 1/k_1 - 1/k$. Then $\hat{g} = \hat{\mathcal{B}}_k \hat{f}$ is k_2-summable in direction d, and $\mathcal{S}_{k_2,d}\,\hat{g} = \mathcal{B}_k(\mathcal{S}_{k_1,d}\,\hat{f})$.*

Proof: Follows immediately from Theorem 23 (p. 80) and the definition of k_1- resp. k_2-summability in direction d. □

On one hand, for $k < 1/2$ the k-sum of formal series is analytic in sectorial regions of opening more than 2π, which forces us to consider such regions on the Riemann surface of the logarithm. On the other hand, statement (c) of the next lemma shows that k-summability does *not distinguish between directions differing by multiples of* 2π.

Lemma 10

(a) *Let $\hat{f}$ be k-summable in direction d, for every $d \in (\alpha,\beta)$, $\alpha < \beta$, and fixed $k > 0$. Then $(\mathcal{S}_{k,d_1}\,\hat{f})(z) = (\mathcal{S}_{k,d_2}\,\hat{f})(z)$ for every $d_1, d_2 \in (\alpha,\beta)$ and every z where both sides are defined.*

(b) *Let $\hat{f}$ be k_j-summable in direction d, $1 \le j \le 2$, with $k_1 > k_2 > 0$ and some fixed d, then $\hat{f}$ is k_1-summable in all directions $\tilde{d}$ with $2\,|d-\tilde{d}| \le \pi\,(1/k_2 - 1/k_1)$, and $(\mathcal{S}_{k_1,\tilde{d}}\,\hat{f})(z) = (\mathcal{S}_{k_2,d}\,\hat{f})(z)$ for every z where both sides are defined.*

(c) *For $\tilde{d} = d + 2\pi$, k-summability of $\hat{f}$ in direction d is equivalent to k-summability of $\hat{f}$ in direction $\tilde{d}$, and $(\mathcal{S}_{k,\tilde{d}}\,\hat{f})(z) = (\mathcal{S}_{k,d}\,\hat{f})(ze^{-2\pi i})$ for every z where both sides are defined.*

Proof:

(a) The function $g = \mathcal{S}\,(\hat{\mathcal{B}}_k \hat{f})$ is holomorphic and of exponential growth not more than k in $S((\alpha+\beta)/2, \beta-\alpha)$. Hence $f = \mathcal{L}_k g \cong_{1/k} \hat{f}$ in a region which, for every $d \in (\alpha, \beta)$, contains a sector of opening larger than π/k and bisecting direction d. So $\mathcal{S}_{k,d}\,\hat{f} = f$, for every such d.

(b) From the definition of k_2-summability in direction d we conclude existence of f with $f(z) \cong_{1/k_2} \hat{f}(z)$ in $G(d, \alpha)$ for some $\alpha > \pi/k_2$. Thus, $g = \mathcal{B}_{k_1} f$ is holomorphic and of exponential growth not more than k_1 in $S(d, \alpha - \pi/k_1)$, and the assumption of k_1-summability in direction d implies that g is holomorphic at the origin. Hence statement (b) follows.

(c) Use Exercise 4 on p. 72.

□

Exercises: Always assume $k > 0$.

1. Show that $\hat{f}(z) = \sum_0^\infty \Gamma(1+n/k)\, z^n$ is k-summable in every direction $d \neq 2j\pi$, $j \in \mathbb{Z}$.

2. Assume that $\hat{f}$ is chosen so that $g = \mathcal{S}\,(\hat{\mathcal{B}}_k \hat{f})$ is a rational function. Let d be so that no poles of g lie on the ray $\arg u = d$, and show that then $\hat{f}$ is k-summable in direction d.

3. Show that $\hat{f}(z) = \sum_0^\infty \Gamma(1+n/k)\, z^n / \Gamma(1+n)$ is k-summable in directions d with $(2j + 1/2)\pi < d < (2j + 3/2)\pi$, $j \in \mathbb{Z}$.

4. For $\hat{f}$ as in Exercise 3, let d be such that $(2j-1/2)\pi < d < (2j+1/2)\pi$, $j \in \mathbb{Z}$. Show that $\hat{f}$ is (resp. is not) k-summable in direction d, if $k \geq 1$ (resp. $0 < k < 1$).

5. Assume $\hat{f}$ so that $g = \mathcal{S}\,(\hat{\mathcal{B}}_k \hat{f})$ is as in (4.2) (p. 63). Show that $\hat{f}$ is k-summable in all directions $d \neq 2j\pi$, $j \in \mathbb{Z}$, and is not k-summable in the remaining directions.

6. Show that $\hat{f}(z) = \sum_0^\infty \Gamma(1+2n)\, z^n / \Gamma(1+n)$
   (a) is 1-summable in every direction $d \in [-\pi, \pi)$ but one (which?).
   (b) is 1/2-summable in every direction d with $(2j + 1/2)\pi < d < (2j + 3/2)\pi, \quad j \in \mathbb{Z}$.

6.3 Algebra Properties

Let $k > 0$ and d be given, and let $\mathbb{E}\,\{z\}_{k,d}$ denote the set of all $\hat{f}$ that are k-summable in direction d. The following theorems are direct consequences from the results on Gevrey asymptotics in Section 4.5, together with the definition of k-summability in direction d.

Theorem 34 *For fixed, but arbitrary, $k > 0$ and d, let $\hat{f}, \hat{g}_1, \hat{g}_2 \in \mathbb{E}\{z\}_{k,d}$ be given. Then we have*

$$\begin{aligned} \hat{g}_1 + \hat{g}_2 \in \mathbb{E}\{z\}_{k,d}, &\qquad \mathcal{S}_{k,d}\,(\hat{g}_1 + \hat{g}_2) = \mathcal{S}_{k,d}\,\hat{g}_1 + \mathcal{S}_{k,d}\,\hat{g}_2, \\ \hat{f}' \in \mathbb{E}\{z\}_{k,d}, &\qquad \mathcal{S}_{k,d}\,(\hat{f}') = \frac{d}{dz}(\mathcal{S}_{k,d}\,\hat{f}), \\ \int_0^z \hat{f}(w)dw \in \mathbb{E}\{z\}_{k,d}, &\qquad \mathcal{S}_{k,d}\left(\int_0^z \hat{f}(w)dw\right) = \int_0^z (\mathcal{S}_{k,d}\,\hat{f})(w)\,dw. \end{aligned}$$

Finally, if p is a natural number, then

$$\hat{f}(z^p) \in \mathbb{E}\{z\}_{pk,d/p}, \qquad \mathcal{S}_{pk,d/p}\,(\hat{f}(z^p)) = (\mathcal{S}_{k,d}\,\hat{f})(z^p).$$

Proof: Follows from Theorems 18, 20 in Section 4.5, and Exercise 2 on p. 72. □

Theorem 35 *Let $\mathbb{E}, \mathbb{F}$ both be Banach spaces, and let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$, $\hat{T} \in \mathcal{L}(\mathbb{E}, \mathbb{F})\{z\}_{k,d}$, $\hat{\alpha} \in \mathbb{C}\{z\}_{k,d}$. Then*

$$\begin{aligned} \hat{T}\,\hat{f} \in \mathbb{F}\{z\}_{k,d}, &\qquad \mathcal{S}_{k,d}\,(\hat{T}\,\hat{f}) = (\mathcal{S}_{k,d}\,\hat{T})\,(\mathcal{S}_{k,d}\,\hat{f}), \\ \hat{\alpha}\,\hat{f} \in \mathbb{F}\{z\}_{k,d}, &\qquad \mathcal{S}_{k,d}\,(\hat{\alpha}\,\hat{f}) = (\mathcal{S}_{k,d}\,\hat{\alpha})\,(\mathcal{S}_{k,d}\,\hat{f}). \end{aligned}$$

Proof: Follows from Theorem 19 (p. 71). □

Theorem 36 *For fixed, but arbitrary, $k > 0$ and d, let $\hat{f}, \hat{g}_1, \hat{g}_2 \in \mathbb{E}\{z\}_{k,d}$ be given. If $\mathbb{E}$ is a Banach algebra, then*

$$\hat{g}_1\,\hat{g}_2 \in \mathbb{E}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{g}_1\,\hat{g}_2) = (\mathcal{S}_{k,d}\,\hat{g}_1)\,(\mathcal{S}_{k,d}\,\hat{g}_2).$$

Moreover, if $\mathbb{E}$ has a unit element and $\hat{f}$ has invertible constant term, then

$$\hat{f}^{-1} \in \mathbb{E}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{f}^{-1}) = (\mathcal{S}_{k,d}\,\hat{f})^{-1},$$

wherever the right-hand side is defined.

Proof: The first statement follows from the previous theorem and the fact that for a Banach algebra $\mathbb{E}$ every series $\hat{g}_1$ can be identified with the element of $\mathcal{L}(\mathbb{E}, \mathbb{F})\{z\}_{k,d}$ corresponding to the mapping $\hat{g}_2 \mapsto \hat{g}_1\,\hat{g}_2$. For the second one, use Exercise 2 to conclude that $(\mathcal{S}_{k,d}\,\hat{f})^{-1}$ is defined on a sectorial region with opening larger than π/k, then apply Theorem 21 (p. 72). □

Roughly speaking, the results in this section may be easily memorized as saying that *that $\mathbb{E}\{z\}_{k,d}$ has the same algebraic properties as $\mathbb{E}\{z\}$, the space of convergent power series.* What these properties are in detail depends on whether $\mathbb{E}$ is an algebra, or has a unit element. In particular, we obtain that $\mathbb{C}\{z\}_{k,d}$ is *a differential algebra, and $\hat{f} \in \mathbb{C}\{z\}_{k,d}$ is invertible if and only if it has a nonzero constant term.*

Exercises:

1. Let $\mathbb{E}$ be a Banach algebra with unit element e. Show that every $a \in \mathbb{E}$ with $\|e - a\| < 1$ is invertible and
$$a^{-1} = \sum_{n=0}^{\infty} (e-a)^n.$$

2. Under the assumptions of the previous exercise, let f be holomorphic in a sectorial region G, and assume that $f_0 = \lim f(z)$ (for $z \to 0$ in G) exists and is invertible. Show that then a sectorial region $\tilde{G} \subset G$ exists, whose opening is smaller than, but arbitrarily close to, that of G, so that for $z \in \tilde{G}$ all values $f(z)$ are invertible.

3. Let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$ have zero constant term. Show that $z^{-1}\hat{f}(z) \in \mathbb{E}\{z\}_{k,d}$, and $\mathcal{S}_{k,d}\,(z^{-1}\hat{f}(z)) = z^{-1}\,(\mathcal{S}_{k,d}\,\hat{f})(z)$.

4. Let $\mathbb{E}$ be a Banach algebra, and let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$ *formally* satisfy the n-th order linear differential equation
$$\sum_{j=0}^{n} a_j(z) y^{(j)} = 0,$$
with coefficients $a_j(z)$ holomorphic near the origin. Show that $f = \mathcal{S}_{k,d}\,(\hat{f})$ then is a solution of the same equation.

5. Let $\mathbb{E}$ be a Banach space whose elements are functions of one or several variables $\omega = (\omega_1, \ldots, \omega_\nu)$ on some domain $\Omega \subset \mathbb{C}^\nu$. Moreover, assume that the norm on $\mathbb{E}$ is such that the mapping $f \mapsto f(\omega)$, for fixed $\omega \in \Omega$, is a continuous linear functional on $\mathbb{E}$; e.g., this is so if $\|f\| = \sup_\Omega |f(\omega)|$. For $\hat{f}(z) = \sum f_n\, z^n \in \mathbb{E}\,[[z]]$ we then have that all coefficients f_n are functions of ω, and we can consider the mapping $\hat{f}(z) \mapsto \hat{f}_\omega(z) = \sum f_n(\omega)\, z^n$ from $\mathbb{E}\,[[z]]$ to $\mathbb{C}\,[[z]]$, for every fixed $\omega \in \Omega$. Show: If $\hat{f}$ is k-summable in direction d, then so is $\hat{f}_\omega$. Hence, the above mapping maps $\mathbb{E}\{z\}_{k,d}$ into $\mathbb{C}\{z\}_{k,d}$, for every ω.

6.4 Definition of k-Summability

In view of Lemma 10 part (c), we will from now on *identify directions d that differ by integer multiples of* 2π, despite the fact that regions still are considered on the Riemann surface of the logarithm. From Lemma 8 part (b) we see that the set of directions d for which some $\hat{f}$ is k-summable is always open. Examples show that the complement of this set can be uncountable;

however, in most applications one encounters series $\hat{f}$ *being k-summable in all directions d but* (after identification modulo 2π) *finitely many directions $d_1, \ldots, d_n$*. Whenever this is so, we simply call $\hat{f}$ *k-summable*. The directions $d_1, \ldots, d_n$ then are called *the singular directions of $\hat{f}$*, provided they occur. For the set of all k-summable series $\hat{f}$ we write $\mathbb{E}\{z\}_k$, and it may sometimes be convenient to define $\mathbb{E}\{z\}_\infty = \mathbb{E}\{z\}$, the set of all convergent series.

It is immediately clear that Theorems 34–36 remain correct if we replace $\mathbb{E}\{z\}_{k,d}$ by $\mathbb{E}\{z\}_k$, and the identities for the respective sums then hold for every but finitely many d (modulo 2π).

Lemma 10 part (a) says that $\mathcal{S}_{k,d}\,\hat{f}$ is independent of d, as d varies in an interval not containing singular directions. At a singular direction, however, the sum will change abruptly, as can be seen from the following proposition.

Proposition 12 *For $\alpha < d_0 < \beta$, $k > 0$, assume that $\hat{f}$ is k-summable in all directions $d \in (\alpha, \beta)$, $d \neq d_0$. For some d_1, d_2 with $\alpha < d_1 < d_0 < d_2 < \beta$, $|d_1 - d_2| < \pi/(2k)$, assume*

$$(\mathcal{S}_{k,d_1}\,\hat{f})(z) = (\mathcal{S}_{k,d_2}\,\hat{f})(z),$$

for all z where both sides are defined. Then $\hat{f}$ is k-summable in direction d_0.

Proof: For $1 \le j \le 2$, the functions $f_j = \mathcal{S}_{k,d_j}\,\hat{f}$ are holomorphic in sectorial regions $G_j = G(d_j, \alpha_j)$ of opening $\alpha_j > \pi/k$. Owing to our assumptions, these sectors overlap on the Riemann surface of the logarithm, and within their intersection the functions are equal. Hence $f = f_1 = f_2$ is holomorphic in $G_1 \cup G_2$, and $f(z) \cong_k \hat{f}(z)$ there. Obviously, $G_1 \cup G_2$ contains a sectorial region of opening larger than π/k and bisecting direction d_0, so $\hat{f}$ is k-summable in direction d_0. □

If a formal power series is k-summable in every direction, then the next proposition shows convergence. In the terminology of summability theory one can therefore say: *Absence of singular rays is a Tauberian condition for k-summability.*

Proposition 13 *Assume $\hat{f} \in \mathbb{E}\{z\}_k$ has no singular direction; in other words, $\hat{f} \in \mathbb{E}\{z\}_{k,d}$ for every d. Then $\hat{f} \in \mathbb{E}\{z\}$, i.e., $\hat{f}(z)$ converges for sufficiently small $|z|$.*

Proof: According to Lemma 10 parts (a), (c), the function $f = \mathcal{S}_{k,d}\,(\hat{f})$ is independent of d and single-valued; i.e., $f(ze^{2\pi i}) = f(z)$ for every z. From Proposition 7 (p. 65) we then obtain $\hat{f} \in \mathbb{E}\{z\}$. □

Suppose $\hat{f} \in \mathbb{E}\{z\}_k$ is given, and d_0 is a singular direction of $\hat{f}$. Then for $g = \mathcal{S}\,(\hat{\mathcal{B}}_k \hat{f}_k)$ two things can happen: Either $g(u)$ is singular for some

u_0 with $\arg u_0 = d_0$, so that holomorphic continuation along $\arg u = d_0$ breaks down at this point; or otherwise, since directions d close to d_0 are not singular, $g(u)$ is holomorphic in a sector $S(d_0, \varepsilon)$, for some $\varepsilon > 0$, but is not of exponential growth at most k along $\arg u = d_0$. This can occur, according to Exercise 5 on p. 102, but only if $g(u)$ is of *infinite exponential* order along $\arg u = d_0$; this can be seen from the following application of Phragmen-Lindelöf's principle:

Proposition 14 *For $k > 0$ and $\alpha < d_0 < \beta$, assume that $\hat{f}$ is k-summable in direction d for every $d \neq d_0$, $d \in (\alpha, \beta)$. Moreover, for some $\tilde{k} > k$ assume that $g = \mathcal{S}\,(\hat{\mathcal{B}}_k \hat{f})$ can be holomorphically continued along* $\arg u = d_0$ *and is of exponential growth at most $\tilde{k}$ along* $\arg u = d_0$. *Then $\hat{f}$ is k-summable in direction d_0.*

Proof: In $S = S((\alpha + \beta)/2, \beta - \alpha)$ we have that $g = \mathcal{S}\,(\hat{\mathcal{B}}_k \hat{f})$, along every ray $\arg u = d$ but for $d = d_0$, is of exponential growth at most k. An application of *Phragmen-Lindelöf's principle* (p. 235) then shows that the same holds for $d = d_0$. □

From Exercise 6 on p. 102 we learn that for fixed d a divergent series $\hat{f}$ may very well be k_1- and k_2-summable in direction d. This changes drastically, if we require the same for all but finitely many directions at the same time, because then it converges:

Theorem 37 *For $k_1 > k_2 > 0$ we have*

$$\mathbb{E}\,\{z\}_{k_2} \cap \mathbb{E}\,\{z\}_{k_1} = \mathbb{E}\,\{z\}_{k_2} \cap \mathbb{E}\,[[z]]_{1/k_1} = \mathbb{E}\,\{z\}.$$

Proof: Trivially $\mathbb{E}\,\{z\} \subset \mathbb{E}\,\{z\}_{k_2} \cap \mathbb{E}\,\{z\}_{k_1} \subset \mathbb{E}\,\{z\}_{k_2} \cap \mathbb{E}\,[[z]]_{1/k_1}$. If $\hat{f} \in \mathbb{E}\,\{z\}_{k_2} \cap \mathbb{E}\,[[z]]_{1/k_1}$, then $\hat{g} = \hat{\mathcal{B}}_{k_2}(\hat{f})$ is entire and of exponential growth at most k everywhere, with $1/k = 1/k_2 - 1/k_1$. Therefore, Proposition 14 implies that $\hat{f}$, with respect to k_2-summability, cannot have any singular directions, hence converges because of Proposition 13 (p. 105). □

The previous theorem can be rephrased as saying that k-summable series of Gevrey order strictly smaller than $1/k$ necessarily converge. Thus, we found another Tauberian condition for k-summability! In fact, the proof shows that being of Gevrey order strictly smaller than $1/k$ is a sufficient condition for absence of singular directions with respect to k-summability. So Proposition 14 is the stronger result!

Exercises:

1. Investigate k-summability of the formal power series $\hat{f}$ in the Exercises in Section 6.2.

2. For $k_1 > k_2 > 0$, let $\hat{f} \in \mathbb{E}\{z\}_{k_2,d} \cap \mathbb{E}[[z]]_{1/k_1}$, for every $d \in (\alpha, \beta)$, $\alpha < \beta$. Prove $\hat{f} \in \mathbb{E}\{z\}_{k_1,d}$ for every $d \in (\alpha - \pi/(2k), \beta + \pi/(2k))$ with $1/k = 1/k_2 - 1/k_1$.

3. For $\operatorname{Re} \lambda > 0$, show that $\hat{f}(z;\lambda) = \sum_0^\infty \Gamma(\lambda + n)\, z^n$ is 1-summable, its 1-sum being the function
$$f(z;\lambda) = z^{-\lambda} \int_0^{\infty(d)} \frac{w^{\lambda-1}}{1-w}\, e^{-w/z}\, dw,$$
for $d \neq 2k\pi$, $k \in \mathbb{Z}$.

4. For complex $\lambda \neq 0, -1, -2, \ldots$ and $\hat{f}(z;\lambda)$ as above, show $\hat{f}(z;\lambda) \in \mathbb{E}\{z\}_1$.

5. For $\hat{f}(z) = \sum f_n z^n \in \mathbb{E}\{z\}_1$ and complex α, β with $\beta \neq 0, -1, \ldots$, show
$$\hat{f}(z;\alpha,\beta) = \sum f_n \frac{(\alpha)_n}{(\beta)_n} z^n \in \mathbb{E}\{z\}_1.$$

6. Show that the *generalized hypergeometric series*
$$F(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^\infty \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}$$
(with $\beta_k \neq 0, -1, -2, \ldots$) is 1-summable for $p = q + 2$, $q \geq 0$.

6.5 General Moment Summability

In Section 5.5 we have introduced general pairs of integral operators with features similar to those of Laplace and Borel operator. Here we shall show that each such pair defines a summability method, and that all methods corresponding to kernels of a fixed order $k > 0$ are *equivalent* in the sense that they sum the same formal power series to the same analytic functions. However, observe that we do not claim that two such kernels of the same order represent the sum on exactly the same sectorial region. Moreover, the two summability methods may be inequivalent for series other than power series.

All the summability methods we consider, aside from some minor modifications, fit into the following classical family of methods:

MOMENT METHODS

Let $e(x)$ be positive and continuous on the positive real axis and asymptotically zero as $x \to \infty$, so that all *moments* $m(n) =$

$\int_0^\infty x^n\, e(x)\, dx$, $n \ge 0$, exist and are positive. A series $\sum x_n$ then is said to be m_e-summable if the power series $x(t) = \sum_0^\infty t^n\, x_n/m(n)$ converges for every $t \in \mathbb{C}$, and

$$\lim_{T\to\infty} \int_0^T e(t)\, x(t)\, dt = \int_0^\infty e(t)\, x(t)\, dt$$

exists.

Applied to formal power series, it is not clear in general whether summability for some $z \neq 0$ implies summability for other values, in particular for such values closer to the origin. Moreover, in applications to ODE, it is more natural not to require convergence of $\sum t^n\, x_n/m(n)$ for every t, but instead be content with a positive radius of convergence plus existence of holomorphic continuation along the positive real axis. In detail, this leads to the following modified definition for summability of formal power series by means of methods defined by kernel functions introduced in Sections 5.5 and 5.6. For further details, see [25].

Moment Summability of Power Series

Let a kernel function $e(z)$ of order $k > 0$, with corresponding integral operator T, be given. We say that a formal power series $\hat{f}(z)$ is T-summable in direction d if the following holds:

(S1) The series $\hat{g} = \hat{T}^- \hat{f}$ has positive radius of convergence.

(S2) For some $\varepsilon > 0$, the function $g = \mathcal{S}\, \hat{g}$ can be holomorphically continued into $S = S(d, \varepsilon)$ and is of exponential growth at most k there.

Obviously, (S1) holds if and only if $\hat{f} \in \mathbb{E}\,[[z]]_{1/k}$. Condition (S2) implies applicability of the integral operator T to g, and we call $f = T\, g$ *the T-sum of $\hat{f}$*, and write $f = \mathcal{S}_{T,d}\, \hat{f}$. The set of all such $\hat{f}$ shall be denoted by $\mathbb{E}\,\{z\}_{T,d}$.

Due to the results of Section 5.5 it is immediately seen that the above summability method is equivalent to k-summability in the following sense:

Theorem 38 *Let an arbitrary kernel function $e(z)$ of order $k > 0$ be given. Then $\mathbb{E}\,\{z\}_{T,d} = \mathbb{E}\,\{z\}_{k,d}$, and for every $\hat{f} \in \mathbb{E}\,\{z\}_{T,d}$ we have $(\mathcal{S}_{T,d}\, \hat{f})(z) = (\mathcal{S}_{k,d}\, \hat{f})(z)$ on some sectorial region of bisecting direction d and opening more than π/k.*

Proof: For $\hat{f} \in \mathbb{E}\,\{z\}_{T,d}$, we conclude from Theorem 27 (p. 91) that $\mathcal{S}_{T,d}\, \hat{f}(z) \cong_{1/k} \hat{f}(z)$ in a sectorial region G with bisecting direction d and opening more than π/k, so by definition $\hat{f} \in \mathbb{E}\,\{z\}_{k,d}$ and $\mathcal{S}_{T,d}\, \hat{f}(z) =$

$\mathcal{S}_{k,d} \hat{f}(z)$ in G. Conversely, $\hat{f} \in \mathbb{E}\{z\}_{k,d}$ implies, owing to Theorem 28 (p. 91), that $g = T^-(\mathcal{S}_{k,d} \hat{f})$ is analytic and of exponential growth not more than k in a sector $S(d, \varepsilon)$, and $g(u) \cong_0 \hat{g}(u) = \hat{T}^- \hat{f}(u)$ there. An asymptotic of Gevrey order zero implies convergence of $\hat{g}$ to g, for small $|u|$. Hence (S1), (S2) follow. □

The equivalence of these summability methods is used in the following exercises:

Exercises: Let $k > 0$ be fixed and observe the following terminology: A sequence $(\lambda_n)_{n \geq 0}$ is called *a summability factor* for k-summability, if for every Banach space $\mathbb{E}$ and every $\sum f_n z^n \in \mathbb{E}\{z\}_k$, we have $\sum \lambda_n f_n z^n \in \mathbb{E}\{z\}_k$.

1. Show that if (λ_n) is a summability factor for k-summability, then necessarily $\sum \lambda_n z^n \in \mathbb{C}\{z\}_k$.
2. Let $e(z)$ be a kernel function of order k, with corresponding moment function $m(u)$. Show that the sequences $(\Gamma(1+n/k)/m(n))$ and $(m(n)/\Gamma(1+n/k))$ both are summability factors for k-summability.
3. Let $e_1(z)$, $e_2(z)$ be kernel functions of the same order k, with corresponding moment functions $m_1(u)$, $m_2(u)$. Show that the sequence $m_1(n)/m_2(n)$ is a summability factor for k-summability.
4. Show that the following sequences are summability factors for k-summability:
   (a) $\lambda_n = \frac{\Gamma(\alpha+n/k)}{\Gamma(\beta+n/k)}$, for $\alpha, \beta > 0$.
   (b) $\lambda_n = \frac{\Gamma(1+s_1 n)(1+s_2 n)}{\Gamma(1+s_3 n)(1+s_4 n)}$, for $s_j \geq 0$, $s_1 + s_2 = s_3 + s_4$.
5. Show that the sequence $\lambda_n = \lambda^n$, for any fixed $\lambda \in \mathbb{C}$, is a summability factor for k-summability.
6. For fixed $p \in \mathbb{N}$, show that the sequence $\lambda_n = 1$ (resp. $= 0$) whenever n is (resp. is not) a multiple of p, is a summability factor for k-summability.
7. For $m \in \mathbb{Z}$, show that the sequence $\lambda_n = (1+n)^m$ is a summability factor for k-summability.
8. Show that the sequence $\lambda_n = 1$ (resp. $= 0$) whenever n is (resp. is not) of the form 2^m with $m \in \mathbb{Z}$, is *not* a summability factor for k-summability.
9. For $s > 0$, show that $(1/\Gamma(1+sn))$ is a summability factor for k-summability if and only if $s \geq 1/k$.

6.6 Factorial Series

In this section, we briefly investigate infinite series of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{(z+1)\cdot\ldots\cdot(z+n)}, \quad b_n \in \mathbb{E}\,. \tag{6.1}$$

The form of the terms requires the variable z to be different from negative integers; for convenience, we shall always restrict z to the right half-plane.

Series of this or a very similar form have frequently been studied in connection with Laplace or Mellin transform, and are usually called *(inverse) factorial series.* A number of authors have used them to represent solutions of ODE, or of *difference equations.* For an application to even more general equations, see *Braaksma and Harris Jr.* [72], or *Gérard and Lutz* [107].

Unlike power series, factorial series converge in half-planes: From the functional equation of the Gamma function (B.9) and (B.13) (p. 231), it is easy to conclude that

$$\frac{n!}{(z+1)\cdot\ldots\cdot(z+n)} = \frac{\Gamma(1+n)\,\Gamma(1+z)}{\Gamma(1+z+n)} \sim \frac{\Gamma(1+z)}{n^z},$$

in the sense that the quotient of the two sides tends to one as $n \to \infty$. Hence, if (6.1) *converges for some* $z = z_0$ in the right half-plane, then the terms necessarily tend to 0. So we obtain that $|b_n| \le n^{\mathrm{Re}\, z_0}\, n!$ for large n. Consequently, we have absolute and locally uniform convergence of (6.1) *for all* z with $\mathrm{Re}\, z > c = 1 + \mathrm{Re}\, z_0$. We also observe that the *rate of convergence of factorial series, in general, is rather slow*, limiting their usefulness for numerical purposes.

It is well understood (see [204], or *Wasow* [281]) how factorial series are related to Laplace integrals, and we here are going to show corresponding relations with k-summability. To do so, we shall make use of the following numbers:

Stirling Numbers

The numbers Γ^n_m, defined by

$$\omega\,(\omega+1)\cdot\ldots\cdot(\omega+n-1) = \sum_{m=1}^{n} \Gamma^n_{n-m}\,\omega^m, \quad n \ge 1,\ \omega \in \mathbb{C},$$

are called *Stirling numbers*, or *factorial coefficients.* For a recursive definition of these numbers and other details, see the exercises below.

To establish the said connection between k-summability and factorial series, we first treat the case of $k = 1$:

Proposition 15 *Let $d \in \mathbb{R}$ be given. For an arbitrary formal power series $\hat{f}(z) = \sum_0^\infty f_n z^n \in \mathbb{E}\{z\}_{1,d}$, let $\omega = r\mathrm{e}^{id}$, $r > 0$, and define*

$$b_n(\omega) = \sum_{m=1}^{n} \omega^m f_m \, \Gamma_{n-m}^n, \quad n \geq 1. \tag{6.2}$$

Then for every sufficiently small $r > 0$, there exists a number $c > 0$ so that the factorial series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n(\omega)}{(\omega/z+1)\cdot \ldots \cdot (\omega/z+n)} \tag{6.3}$$

converges absolutely for $\mathrm{Re}\,(\omega/z) > c$. *The function $f(z)$ then does not depend upon r, and*

$$f_0 + f(z) = (\mathcal{S}_{1,d}\, \hat{f})(z), \tag{6.4}$$

for such z where both sides give sense.

Proof: According to our assumptions, the function $g = \mathcal{S}\,(\hat{\mathcal{B}}_1 \hat{f})$ is analytic in $G = D(0,\rho) \cup S(d,\varepsilon)$, for sufficiently small $\rho, \varepsilon > 0$. With r small enough, the function $g(u\omega)$ is holomorphic in the region defined by $|1-\mathrm{e}^{-u}| < 1$, and continuous up to its boundary. In other words, the function $h(t) = g(-\omega \log[1-t])$ is analytic in the unit disc, and continuous along its boundary except for a singularity at $t = 1$. In the sector $S(d,\varepsilon)$, the function $g(u)$ is of exponential growth at most one. We may assume without loss of generality that $\varepsilon \leq \pi/2$, so that this fact can be expressed as $|g(u)| \leq c|\exp[K\,u\,\mathrm{e}^{-id}]|$ in G, for sufficiently large $c, K > 0$. This then implies

$$|h(t)| \leq \frac{c}{|1-t|^{rK}}, \quad |t| \leq 1, \tag{6.5}$$

and we may take $0 < r < 1/K$, so that $h(t)$ has an integrable singularity at $t = 1$. As follows from Exercise 4 below, the coefficients h_n of the power series expansion of $h(t)$ about the origin are equal to $b_n(\omega)/n!$, with $b_n(\omega)$ given by (6.2), for $n \geq 1$, while $h_0 = f_0$. Using Cauchy's Formula and (6.5), we find for the remainder term $h_N(t) = \sum_{n=N}^\infty h_n t^n$:

$$|h_N(t)| = \left| \frac{t^N}{2\pi i} \oint_{|u|=1} \frac{h(u)\,du}{u^N\,(u-t)} \right| \leq \frac{c}{2\pi|1-t|} \int_0^{2\pi} \frac{d\phi}{|1-\mathrm{e}^{i\phi}|^{Kr}},$$

for $0 \leq t < 1$. In the original variable u, this implies

$$g(u) = f_0 + \sum_{n=1}^{\infty} \frac{b_n(\omega)}{n!}\,(1-\mathrm{e}^{-u/\omega})^n, \;\; |g_N(u)| = |h_N(1-\mathrm{e}^{-u/\omega})| \leq \tilde{c}|\mathrm{e}^{u/\omega}|,$$

for $\arg u = d$. Hence, by means of *Lebesgue's dominated convergence theorem*, we conclude that we may insert the expansion for $g(u)$ into Laplace transform and integrate termwise. Using the exercise below to find the Laplace transform for $(1-e^{-u/\omega})^n$, we obtain (6.4), completing the proof. □

The above proposition implies for $f_n = \delta_{nm}$, $m \in \mathbb{N}$, that

$$z^m = \omega^m \sum_{n=m}^{\infty} \frac{\Gamma_{n-m}^n}{(\omega/z+1)\cdot \ldots \cdot (\omega/z+n)},$$

where ω can be any complex number. An analysis of the proof shows that this series converges whenever ω/z is is the right half-plane. This implies that for an arbitrary formal power series $\hat{f}(z) = \sum_{m=1}^{\infty} f_m\, z^m$ we may replace z^m by the above expansion and formally interchange the two sums in order to obtain a *formal factorial series*. Thus, the above proposition may be viewed as saying that for 1-summable series this process leads to a *convergent* factorial series. Since $\hat{f} \in \mathbb{E}\{z\}_{1,d}$ implies $\hat{f} \in \mathbb{E}\{z\}_{1,\tau}$, for τ sufficiently close to d, we obtain that (6.3) remains convergent for $\omega = re^{i\tau}$, with τ as above, and $0 < r \le r_0(\tau)$. This observation leads to the following converse of Proposition 15, showing that 1-summability and summation in form of a factorial series are equivalent.

Proposition 16 *Let $d \in \mathbb{R}$ be given. For an arbitrary formal power series $\hat{f}(z) = \sum_0^{\infty} f_n\, z^n$, let $\omega = re^{id}$, $r > 0$, and define $b_n(\omega)$ as in* (6.2)*. For $\alpha < d < \beta$, assume existence of $r_0 = r_0(d) > 0$ so that for $0 < r < r_0$ the factorial series* (6.3) *converges absolutely for* $\mathrm{Re}\,(\omega/z) > c$*, with suitably large $c = c(r) \ge 0$. Then, $\hat{f} \in \mathbb{E}\{z\}_{1,d}$ for every d as above.*

Proof: Observe that absolute convergence of (6.3) implies $f(z) \to 0$ as $z \to 0$, uniformly in the region $\mathrm{Re}\,(\omega/z) \ge c+\varepsilon$, for every $\varepsilon > 0$. Suppose that we had shown f to be independent of ω – then all that were left to show would be that $g = \mathcal{B}_1 f$ is holomorphic near the origin. Both, however, can be obtained at once as follows:

For fixed ω, we use (5.4) (p. 82) to compute $g(u) = g(u;\omega) = (\mathcal{B}_1 f)(u)$, for $u = xe^{id}$, $x > 0$. Moreover, we can interchange summation and limit $y \to \infty$ to obtain $g(u) = \sum_{n=1}^{\infty} b_n(\omega)\,(1-e^{-u/\omega})^n/n!$, for u as above. This is a power series in $t = 1-e^{-u/\omega}$, and consequently we obtain convergence in some disc $|t| < \rho$. This shows holomorphy of g at the origin. Re-expanding the above series as a power series in u and using (6.2) then shows that $g = \mathcal{S}\,(\hat{\mathcal{B}}_1 \hat{f})$, so g does not depend upon ω (and then so does f), completing the proof. □

While above we treated the case of $k = 1$, we shall now briefly discuss a more general situation of *rational* $k = p/q$, with $p, q \in \mathbb{N}$ being co-prime. As we shall see later, this is good enough when dealing with systems of ODE, since there all “levels” will indeed be rational. So assume for such k that $\hat{f}(z) = \sum f_n\, z^n$ is k-summable in direction d, and let $g(u)$ stand for the sum of $\hat{\mathcal{B}}_k \hat{f}$, and $f(z)$ for the k-sum in direction d of $\hat{f}(z)$. Perhaps, the first idea that comes to mind is to observe that then $f(z^{1/k})$ is the Laplace transform of index one of $g(u^{1/k})$, so that one might think of applying the above propositions. However, $g(u^{1/k})$ will in general not be analytic at the origin, but instead has a branch-point of order p, or in other words, has a representation in terms of a power series in $u^{1/p}$. To remedy this, we form the series

$$\hat{f}_j(z) = \sum_{n=0}^{\infty} f_{pn+j}\, z^{pn}, \quad 0 \le j \le p-1, \tag{6.6}$$

so that $\hat{f}(z) = \sum_{j=0}^{p-1} z^j \hat{f}_j(z)$. According to Exercise 6 below, the series $\hat{f}_j(z)$ are going to be k-summable in direction d if and only if the original series $\hat{f}(z)$ is k-summable *in all directions of the form* $d + 2j\pi/p$, $0 \le j \le p-1$. Then we can indeed apply the above propositions to each of the series $\hat{f}_j(z^{1/k})$, obtaining representations of their sums as convergent factorial series in the variable ω/z^k. Combining these representations, one then finds a corresponding formula for the sum of $\hat{f}(z)$. Since we are not going to use this, we shall not work out the details of this approach here, but mention that this approach has already been taken by *Nevanlinna* [201].

Here we have restricted ourselves to investigating *convergence* of factorial series. In the context of difference equations, *divergent* factorial series also arise, and their summability properties may be investigated. For partial results in this direction, see [28, 48]. Instead of factorial series, several authors have given representations of solutions of ODE in terms of higher transcendental functions. These series have the advantage of converging "globally" and can be used in the context of the *central connection problem.* We here mention *Schmidt* [244, 245], *Kurth and Schmidt* [165], *Dunster and Lutz* [90], and *Dunster, Lutz, and Schäfke* [91].

Exercises:

1. Setting $\Gamma^n_m = 0$ for $m \ge n$, show the recursion
$$\Gamma^{n+1}_m = n\,\Gamma^n_{m-1} + \Gamma^n_m, \quad n \ge m \ge 0.$$

2. Let $g(u)$ be analytic near the origin. For $n \ge 1$, show the existence of numbers b_{nm}, independent of g, such that
$$(1-t)^n \frac{d^n}{dt^n} g\left(-\log[1-t]\right) = \sum_{m=0}^{n-1} b_{nm}\, g^{(n-m)}\left(-\log[1-t]\right).$$
Applying this to $g(u) = e^{zu}$, conclude $b_{nm} = \Gamma^n_m$.

3. Let $g(u) = \sum_0^\infty g_n\, u^n$, $|u| < \rho$, with $\rho > 0$. Conclude that then $g(-\log[1-t]) = \sum_0^\infty h_n\, t^n$, $|t| < \tilde{\rho}$, with $h_0 = g_0$ and
$$h_n\, n! = \sum_{m=1}^{n} \Gamma^n_{n-m} g_m\, m!, \qquad n \ge 1.$$

4. Under the assumptions of Proposition 15, use the previous exercise to show for $n \in \mathbb{N}$ that the coefficients h_n, defined in the proof, are equal to $b_n(\omega)/n!$, with $b_n(\omega)$ given by (6.2).

5. Show that the Laplace transform of order 1 of $(1 - \mathrm{e}^{-u/\omega})^n$ equals $[(\omega/z + 1) \cdot \ldots \cdot (\omega/z + n)]^{-1}\, n!$.

6. Given $\hat{f}(z) \in \mathbb{E}\,[[z]]$, let $\hat{f}_j(z)$ be defined as in (6.6). Show that

$$\hat{f}_j(z) = p\, z^{-j} \sum_{\mu=0}^{p-1} \mathrm{e}^{-2\pi i j\mu/p}\, \hat{f}(z\mathrm{e}^{2\pi i\mu/p}), \quad 0 \le j \le p-1,$$

and use this to show that all $\hat{f}_j(z)$ are k-summable in direction d if and only if $\hat{f}(z)$ is k-summable in all directions of the form $d + 2j\pi/p$, $0 \le j \le p-1$.

7. Using the Stirling numbers, show that every formal power series with a zero constant term can be formally rewritten as a, usually divergent, factorial series of the form (6.3), and vice versa.

7

Cauchy-Heine Transform

Let $\psi(w)$ be a continuous function for w on the straight line segment from 0 to a point $a \neq 0$. Then the function

$$f(z) = \frac{1}{2\pi i} \int_0^a \frac{\psi(w)}{w - z}\, dw$$

obviously is holomorphic for z in the complex plane with a cut from 0 to a, but f in general will be singular at points on this cut. If ψ is holomorphic, at least at points strictly between 0 and a, then one can use Cauchy's integral formula to see that f is holomorphic at these points, too. At the endpoints, however, f will be singular, even if ψ is analytic there; for this, see the exercises at the end of the first section. For our purposes, it will be important to assume that ψ, for $w \to 0$, decreases faster than arbitrary powers of w, since then we shall see that f will have an asymptotic power series expansion at the origin. We shall even show that this expansion is of Gevrey order s, provided that $\psi(w) \cong_s \hat{0}$. Hence integrals of the above type provide an excellent tool for producing examples of functions with asymptotic expansions, or even of series that are k-summable in certain directions. Much more can be done, however: For arbitrary functions, analytic in a sectorial region and having an asymptotic expansion at the origin, we shall obtain a representation that is the analogue to Cauchy's formula for functions analytic at the origin. As a special case, we shall obtain a very useful characterization of such functions f that are the sums of k-summable series in some direction d. In other words, we shall *characterize the image of the operator* $\mathcal{S}_{k,d}$, for $k > 0$ and $d \in \mathbb{R}$. As another application of this representation formula, we shall prove several *decomposition* *theorems.* For example, we shall show that a divergent k-summable series can be decomposed into finitely many such series that all have exactly one singular direction. An even more important decomposition theorem for multisummable series will be proven in Section 10.4.

7.1 Definition and Basic Properties

Let $s > 0$ and a sectorial region $G = G(d, \alpha)$ be given. We shall write $\boldsymbol{A}_{s,0}(G, \mathbb{E})$ for the set of $\psi \in \boldsymbol{A}_s(G, \mathbb{E})$ with $J(\psi) = \hat{0}$. From Proposition 11 (p. 75) we conclude that for $\alpha > s\pi$ the space $\boldsymbol{A}_{s,0}(G, \mathbb{E})$ only contains the zero function. Hence we may restrict our discussion to regions with $\alpha \leq s\pi$, but everything we say will be trivially correct in other cases, too.

Let $\psi \in \boldsymbol{A}_{s,0}(G, \mathbb{E})$ and fix $a \in G$. Then the function

$$f(z) = (\mathcal{CH}_a\,\psi)(z) = \frac{1}{2\pi i} \int_0^a \frac{\psi(w)}{w - z}\, dw$$

will be called *Cauchy-Heine transform* of $\psi(w)$. Clearly, $f(z)$ is holomorphic for z, on the Riemann surface of the logarithm, in the sector $S = \{z : \arg a < \arg z < 2\pi + \arg a\}$, and vanishes as $z \to \infty$. The function $f(z)$ even is holomorphic at ∞, if we consider z in the complex plane instead of the Riemann surface, but that is of no importance right now. By deforming the path of integration, we can holomorphically continue $f(z)$ into the region $\tilde{G} = G_a \cup S \cup e^{2\pi i} G_a$, where G_a denotes G with points xa $(x \geq 1)$ deleted, and $e^{2\pi i} G_a$ stands for the "same" region as G_a on the next sheet of the Riemann surface. Hence $\tilde{G}$ is a sectorial region of bisecting direction $\tilde{d} = d + \pi$ and opening $\tilde{\alpha} = \alpha + 2\pi$.

Proposition 17 *Let $G, \psi, \tilde{G}, f$ be as above. Then $f(z) \cong_s \hat{f}(z)$ in $\tilde{G}$, with $\hat{f}(z) = \sum_0^\infty f_n\, z^n$ given by*

$$f_n = \frac{1}{2\pi i} \int_0^a \frac{\psi(w)}{w^{n+1}}\, dw, \quad n \geq 0. \tag{7.1}$$

Moreover, if $z \in G$ and $|z| < |a|$, hence both z and $z e^{2\pi i}$ in $\tilde{G}$, then

$$f(z) - f(z e^{2\pi i}) = \psi(z). \tag{7.2}$$

Proof: We have $(w - z)^{-1} = z^N w^{-N} (w - z)^{-1} + \sum_{n=0}^{N-1} z^n w^{-n-1}$ for $N \geq 0$. Hence, with f_n as in (7.1), we find

$$r_f(z, N) = \frac{1}{2\pi i} \int_0^a \frac{\psi(w)}{w^N (w - z)}\, dw, \quad z \in \tilde{G},$$

if we choose the path of integration so that $w - z \neq 0$. Let $\bar{S}$ be a closed subsector of $\tilde{G}$. In case its opening is 2π or more, we may split it into finitely many closed sectors of smaller opening; thus, we may assume the opening to be strictly less than 2π. For such a sector, we can choose a path of integration from 0 to a, independent of $z \in \bar{S}$ and so that $c = c(\bar{S}) > 0$ exists for which $|w - z| \ge c|w|$, for every w on the path and every $z \in \bar{S}$. Since $\psi \in \boldsymbol{A}_{s,0}(G, \mathbb{E})$, we have for sufficiently large $\tilde{c}, K > 0$, independent of w, that $\|w^{-N}\psi(w)\| \le \tilde{c}\, K^N \Gamma(1 + sN)$, for every $N \ge 0$ and every w on the path of integration. This implies, with L denoting the length of the path of integration, $\|r_f(z, N)\| \le c^{-1}\, \tilde{c}\, L\, (2\pi)^{-1}\, K^{N+1}\, \Gamma(1 + [N + 1]s)$, for every $N \ge 0$ and $z \in \bar{S}$. From this follows $f(z) \cong_s \hat{f}(z)$ in $\tilde{G}$. To prove (7.2), we observe

$$f(z) - f(z e^{2\pi i}) = \frac{1}{2\pi i} \oint_\gamma \frac{\psi(w)}{w - z}\, dw$$

with a closed path of positive orientation around z, and use Cauchy's formula. □

Remark 9: Under the assumptions of Proposition 17, and setting $k = 1/s$, we see from the definition on p. 100 that $\hat{f}(z)$ is k-summable in every direction $\hat{d}$ with $2\,|\hat{d} - d - \pi| < \alpha + \pi(2 - s)$. Since α may be arbitrarily small, it may happen for $0 < k < 1/2$ that there is no such $\hat{d}$. This is another reason why values of k below, or even equal to, $1/2$ are special in our theory. Moreover, it is worthwhile to observe for later applications that for $\psi \in \boldsymbol{A}_{s,0}(G, \mathbb{E})$ and $a, b \in G$, the function $\mathcal{CH}_a\, \psi - \mathcal{CH}_b\, \psi$ is *holomorphic at the origin.* ◇

The formal series $\hat{f}$ in Proposition 17 will sometimes be denoted as $\widehat{\mathcal{CH}}_a\, \psi$ and named *formal Cauchy-Heine transform* of ψ.

Exercises: In the following exercises we shall compute the Cauchy-Heine transform of functions ψ that are analytic near the origin; this will not be used later on, but is of interest in its own right.

1. For fixed $a \ne 0$ and $n \ge 0$, show

$$\int_0^a \frac{w^n\, dw}{w - z} = z^n \log(1 - a/z) + \sum_{m=1}^{n} \frac{a^m\, z^{n-m}}{m},$$

for $|z| > |a|$, and then conclude by holomorphy that the formula stays valid for z in the complex plane with a "cut" from 0 to a.

2. For ψ continuous along the line segment from 0 to a and holomorphic in $D(0, \varepsilon)$, for some $\varepsilon > 0$, show for $|z| < \varepsilon$, $\arg z \ne \arg a$:

$$\int_0^a \frac{\psi(w)\, dw}{w - z} = \psi(z) \log(1 - a/z) + \phi(z),$$

with a function $\phi(z)$ which is analytic near the origin.

7.2 Normal Coverings

We frequently consider sectors with openings larger than 2π and consequently have to work on the Riemann surface of the logarithm. Nonetheless, we have seen in Chapter 6 that for k-summability we should identify directions d that differ by integer multiples of 2π, hence should think of rays $\arg z = d$ in the complex plane. Quite similarly, the notion of a *normal covering* to be defined is best understood if pictured in the complex plane:

For a natural number m, let $d_0 < d_1 < \ldots < d_{m-1} < d_0 + 2\pi = d_m$ be given directions. Moreover, let $\alpha_j > 0$, $j = 0, \ldots, m-1$, be so that with $\alpha_m = \alpha_0$ we have $d_j - \alpha_j/2 < d_{j-1} + \alpha_{j-1}/2$, $1 \le j \le m$. If this is so, and $\rho > 0$ is arbitrarily given, then we say that the sectors $S_j = S(d_j, \alpha_j, \rho)$, $0 \le j \le m$, form *a normal covering*. One should observe that on the Riemann surface S_m is directly above S_0. One can also define d_j and α_j, for arbitrary integers j, so that $d_{j+m} = d_j + 2\pi$, $\alpha_{j+m} = \alpha_j$ for every j; then the sectors $S_j = S(d_j, \alpha_j, \rho)$ cover the whole Riemann surface, and $m+1$ consecutive ones always are a normal covering.

Using this notion of normal covering, we now prove the key result for what is to follow:

Theorem 39 *Let $S_j = S(d_j, \alpha_j, \rho)$, $0 \le j \le m$, be a normal covering. For $k > 0$, $s = 1/k$, and $1 \le j \le m$, let $\psi_j \in \boldsymbol{A}_{s,0}(S_{j-1} \cap S_j, \mathbb{E})$ and choose $a_j \in S_{j-1} \cap S_j$. For $0 < \tilde\rho \le |a_j|$ $(1 \le j \le m)$, let $\tilde S_j = S(d_j, \alpha_j, \tilde\rho)$, $0 \le j \le m$, and define*

$$f_j(z) = \sum_{\mu=1}^{j} (\mathcal{CH}_{a_\mu}\, \psi_\mu)(z) + \sum_{\mu=j+1}^{m} (\mathcal{CH}_{a_\mu}\, \psi_\mu)(z e^{2\pi i})$$

for $z \in \tilde S_j$, $0 \le j \le m$, interpreting empty sums as zero. Then we have $f_j \in \boldsymbol{A}_s(\tilde S_j, \mathbb{E})$, with $J(f_j) = \sum_{\mu=1}^m \widehat{\mathcal{CH}}_{a_\mu}\, \psi_\mu$. Hence in particular, $J(f_j)$ is independent of j. Moreover, $f_j(z) - f_{j-1}(z) = \psi_j(z)$, $1 \le j \le m$, $f_m(z e^{2\pi i}) = f_0(z)$.

Proof: For each μ, $\mathcal{CH}_{a_\mu}\, \psi_\mu$ is holomorphic for $d_\mu - \alpha_\mu/2 < \arg z < 2\pi + d_{\mu-1} + \alpha_{\mu-1}/2$, $|z| < \tilde\rho$. So one easily checks for $1 \le \mu \le j$, resp. $j+1 \le \mu \le m$, that $(\mathcal{CH}_{a_\mu}\, \psi_\mu)(z)$, resp. $(\mathcal{CH}_{a_\mu}\, \psi_\mu)(z e^{2\pi i})$, are holomorphic in the sectors $\tilde S_j$, $0 \le j \le m$. The proof then is easily completed, using Proposition 17. □

Exercises:

1. Check that the above theorem for $m = 1$ coincides with Proposition 17.

2. Given a normal covering, try to define what one might call *a finer normal covering*, and check that for any two normal coverings one can always consider *a common refinement*.

7.3 Decomposition Theorems

For $\psi \in \boldsymbol{A}_{s,0}(G, \mathbb{E})$, the coefficients of $\hat{f} = \widehat{\mathcal{CH}}_a\, \psi$, for arbitrary $a \in S$, have the form of a *moment sequence*, so that $\hat{f}$ appears, at first glance, to be rather special. The following theorem shows, however, that arbitrary $\hat{f} \in \mathbb{C}\,[[z]]_s$ admit a decomposition into a convergent series plus finitely many formal Cauchy-Heine transforms:

Theorem 40 *For $s > 0$ and $\hat{f} \in \mathbb{E}\,[[z]]$, we have $\hat{f} \in \mathbb{E}\,[[z]]_s$ if and only if for every normal covering $S_j = S(d_j, \alpha_j, \rho)$ with $0 < \alpha_j \le s\pi$, $0 \le j \le m$, there exist $\psi_j \in \boldsymbol{A}_{s,0}(S_{j-1} \cap S_j, \mathbb{E})$, $1 \le j \le m$, so that for arbitrarily chosen $a_j \in S_{j-1} \cap S_j$ we have*

$$\hat{f} = \hat{f}_0 + \sum_{j=1}^{m} \widehat{\mathcal{CH}}_{a_j}\, \psi_j, \quad \textit{with } \hat{f}_0 \in \mathbb{E}\,\{z\}. \tag{7.3}$$

Proof: Let $\hat{f} \in \mathbb{C}\,[[z]]_s$, and let S_j be as stated, $0 \le j \le m$. Then Proposition 10 (p. 73) implies existence of $\tilde{f}_j \in \boldsymbol{A}_s(S_j, \mathbb{E})$ with $J(\tilde{f}_j) = \hat{f}$, for $j = 0, \ldots, m-1$. Defining $\tilde{f}_m(z) = \tilde{f}_0(z\mathrm{e}^{-2\pi i})$, $z \in S_m$, the same holds for $j = m$. Let $\psi_j(z) = \tilde{f}_j(z) - \tilde{f}_{j-1}(z)$, $z \in S_{j-1} \cap S_j$; then $\psi_j \in \boldsymbol{A}_{s,0}(S_{j-1} \cap S_j, \mathbb{E})$, $1 \le j \le m$. With f_j as in Theorem 39, we then see that $f(z) = \tilde{f}_j(z) - f_j(z)$ is independent of j, and holomorphic and single-valued in a punctured neighborhood of the origin. Moreover, f has an asymptotic expansion there; hence Proposition 7 (p. 65) implies holomorphy of f at the origin. Applying J to $\tilde{f}_j(z) = f(z) + f_j(z)$, we conclude (7.3) with convergent $\hat{f}_0 = J(f)$. The converse statement follows directly from Proposition 17. □

The above theorem and its proof remain correct if we allow normal coverings of possibly larger openings provided that $f_j \in \boldsymbol{A}_s(S_j, \mathbb{E})$ with $J(f_j) = \hat{f}$, $0 \le j \le m-1$, still exist. This will be used in the proof of the following characterisation of k-summable power series:

Corollary to Theorem 40 *Let $s > 0$, $d \in \mathbb{R}$ and $\hat{f} \in \mathbb{E}\,[[z]]$ be given. Then the following holds:*

(a) *For $k = 1/s \ge 1/2$, k-summability of $\hat{f}$ in direction d is equivalent to the existence of a decomposition of $\hat{f}$ as in Theorem 40, with*

$$|d + \pi - \arg a_j| \le \pi(1 - s/2), \quad 1 \le j \le m.$$

(b) *For $k = 1/s \le 1/2$, k-summability of $\hat{f}$ in direction d is equivalent to the existence of a decomposition of $\hat{f}$ as in Theorem 40, with $m = 1$ and*

$$S_0 \cap S_1 = S(d+\pi, \alpha, \rho), \quad \alpha > (s-2)\pi.$$

Proof: Let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$ and $k \ge 1/2$, and take a normal covering S_j, $j = 0, \ldots, m$ as follows: The sector S_0 has bisecting direction d and opening larger than $s\pi$ and is so that $f_0 = \mathcal{S}_{k,d}\,\hat{f}$ is holomorphic in S_0. The openings of the other sectors are at most $s\pi$. Since we are free to choose any $a_j \in S_{j-1} \cap S_j$, it is easy to see that we can make the inequality in statement (a) hold. Thus, we obtain a decomposition as in Theorem 40. In case $0 < k \le 1/2$, we have that $\hat{f} \in \mathbb{C}\{z\}_{k,d}$ implies $f = \mathcal{S}_{k,d}\,\hat{f} \in \boldsymbol{A}_s(S_0, \mathbb{E})$, $S_0 = S(d, \tilde{\alpha}, \rho)$, $\tilde{\alpha} > s\pi$. With $S_1 = S(d+2\pi, \tilde{\alpha}, \rho)$ we see that S_0 and S_1 are already a normal covering, and $S_0 \cap S_1$ is a sector of opening larger than $\pi(s-2)$ and bisecting direction $d+\pi$. So one direction of the proof in both cases is completed. The opposite directions, however, follow from Remark 9 (p. 117). □

We now show that k-summable series can always be decomposed as a sum of such series with just one singular direction:

Theorem 41 *Let $k > 0$ and $\hat{f} \in \mathbb{C}\{z\}_k$ be given, and let $\hat{f}$ have $m \ge 2$ singular directions in any half-open interval of length 2π. Then*

$$\hat{f} = \hat{f}_1 + \ldots + \hat{f}_m,$$

where each $\hat{f}_j \in \mathbb{C}\{z\}_k$ has exactly one singular direction.

Proof: Let the singular directions of $\hat{f}$ be $0 < d_1 < \ldots < d_m \le 2\pi$, and define $d_0 = d_m - 2\pi$. For $1 \le j \le m$, define $f_j = \mathcal{S}_{k,d}\,\hat{f}$, for $d_{j-1} < d < d_j$. Then f_j is holomorphic in a sectorial region G_j with bisecting direction $(d_{j-1}+d_j)/2$ and opening $d_j - d_{j-1} + \pi/k$. Moreover, $f_j(z) \cong_{1/k} \hat{f}(z)$ in G_j. With $G_0 = G_m e^{-2\pi i}$ and $f_0(z) = f_m(z e^{2\pi i})$ in G_0, define $\psi_j(z) = f_j(z) - f_{j-1}(z)$ in $G_{j-1} \cap G_j$, $1 \le j \le m$. For arbitrary $a_j \in G_{j-1} \cap G_j$, Remark 9 (p. 117) implies that $\widehat{\mathcal{CH}}_{a_j}\,\psi_j$ is k-summable in all directions but d_{j-1} modulo 2π. As in the proof of Theorem 40 one can show that $\hat{f} - \sum_{j=1}^m \widehat{\mathcal{CH}}_{a_j}\,\psi_j = \hat{f}_0$ converges; hence is k-summable in every direction. Defining $\hat{f}_j = \widehat{\mathcal{CH}}_{a_j}\,\psi_j$, $1 \le j \le m-1$, and $\hat{f}_m = \hat{f}_0 + \widehat{\mathcal{CH}}_{a_m}\,\psi_m$, the proof is completed. □

Exercises:

1. Show that if $\hat{f} \in \mathbb{E}\{z\}_k$ has exactly one singular ray d_0, then $\hat{f}(z^2)$ is in $\mathbb{E}\{z\}_{2k}$ and has exactly two singular rays $d_0/2$ and $\pi + d_0/2$.

2. Let $\hat{f}(z) = \sum_0^\infty \Gamma(1+n/k)\, z^{2n}$, $k > 0$. Show $\hat{f} \in \mathbb{E}\{z\}_{2k}$, determine the singluar rays of $\hat{f}$, and explicitly write $\hat{f}$ as a sum of formal series which each have one singular ray.

3. Choose $\hat{f} \in \mathbb{C}[[z]]_s$ so that $g = \mathcal{S}(\hat{\mathcal{B}}_{1/s}\hat{f})$ is a rational function. Show that a decomposition of $\hat{f}$ into series, having exactly one singular ray each, can be achieved through a partial fraction decomposition of g.

7.4 Functions with a Gevrey Asymptotic

The following result can be thought of as a generalization of the well-known characterization of removable singularities. As we shall see in the exercises below, it is very useful in showing that some functions have a Gevrey asymptotic.

Proposition 18 *Let $s > 0$, any sector S and any function f, holomorphic in S, be given. Then $f \in \boldsymbol{A}_s(S, \mathbb{E})$ is equivalent to the existence of a normal covering $S_0, \ldots, S_m$, with $S_0 = S$, and functions f_j, holomorphic in S_j, $0 \le j \le m$, with $f_0 = f$ and $f_m(z) = f_0(z\mathrm{e}^{-2\pi i})$, $z \in S_m$, so that all f_j are bounded at the origin, and*

$$f_{k-1}(z) - f_k(z) \in \boldsymbol{A}_{s,0}(S_{k-1} \cap S_k, \mathbb{E}), \quad 1 \le k \le m.$$

Proof: If $f \in \boldsymbol{A}_s(S, \mathbb{E})$, let $\hat{f} = J(f)$, and choose any normal covering $S_0, \ldots, S_m$ so that $S_0 = S$ and the opening of $S_1, \ldots, S_{m-1}$ is less than or equal to $s\pi$. Then Proposition 10 (p. 73) shows existence of $f_j \in \boldsymbol{A}_s(S_j, \mathbb{E})$ with $J(f_j) = \hat{f}$, $1 \le j \le m-1$, and with $f_m(z) = f_0(z\mathrm{e}^{-2\pi i})$ we obtain $f_{k-1}(z) - f_k(z) \in \boldsymbol{A}_{s,0}(S_{k-1} \cap S_k, \mathbb{E})$, $1 \le k \le m$. Conversely, if S_j and f_j are as stated, define $\psi_j = f_j - f_{j-1}$, $1 \le j \le m$, and (with $a_\mu \in S_{\mu-1} \cap S_\mu$ and $\tilde{S}_j$ as in Theorem 39)

$$g_j(z) = \sum_{\mu=1}^{j} (\mathcal{C}\mathcal{H}_{a_\mu}\, \psi_\mu)(z) + \sum_{\mu=j+1}^{m} (\mathcal{C}\mathcal{H}_{a_\mu}\, \psi_\mu)(z\mathrm{e}^{2\pi i})$$

for $z \in \tilde{S}_j$ and $0 \le j \le m$. Then Theorem 39 shows $g_j \in \boldsymbol{A}_s(\tilde{S}_j, \mathbb{E})$ and $J(g_j) = \hat{g}$, independent of j. Moreover, $h(z) = f_j(z) - g_j(z)$ is also independent of j, and single-valued and bounded at the origin. Therefore, the origin is a removable singularity of h, and consequently $f_j = h + g_j \in \boldsymbol{A}_s(\tilde{S}_j, \mathbb{E})$, and even in $\boldsymbol{A}_s(S_j, \mathbb{E})$, in view of Exercise 1 on p. 72, because $\tilde{S}_j$ and S_j only differ in their radius. □

Let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$ be given. Then, there exists a sector S of opening larger than π/k in which $\mathcal{S}_{k,d}\, \hat{f}(z) \cong_{1/k} \hat{f}(z)$ holds. Using the above proposition, one can therefore *characterize the image of the operator* $\mathcal{S}_{k,d}$. This will be done in detail in the context of *multisummable series* later on.

Exercises: Throughout the following exercises, let λ be any complex number, let $p(z)$ be a polynomial of degree $r \geq 1$ and highest coefficient one, and let $g(u)$ be holomorphic (and single-valued) for $0 \leq |u| < \rho$.

1. For $j = 0, \ldots, r$, define

$$f_j(1/z) = z^\lambda e^{p(z)} \int_{\infty(2j\pi/r)}^{z} u^{-\lambda} e^{-p(u)} g(1/u)\, du^r, \quad 0 < |z| < \rho,$$

   integrating from ∞ along the line $\arg u = 2j\pi/r$ to some arbitrarily chosen point $z_{0,j}$, $|z_{0,j}|^{-1} < \rho$, and then to z. Show that each f_j is holomorphic for $0 < |z| < \rho$, and $f_j(1/z) - f_{j-1}(1/z) = c_j z^\lambda e^{p(z)}$, $j = 1, \ldots, r$, with $c_j = \int_{\gamma_j} u^{-\lambda} e^{-p(u)} g(1/u)\, du$, where γ_j is a path from ∞ along $\arg u = 2j\pi/r$ to $z_{0,j}$, then to $z_{0,j-1}$, and back to ∞ along $\arg u = 2(j-1)\pi/r$.

2. For $z \in S_j = S(2j\pi/r, 3\pi/r, \rho)$ and $j = 0, \ldots, r$, show that $f_j(z)$ is bounded at the origin.

3. Show

$$\begin{aligned} &f_{j-1}(z) - f_j(z) \in \boldsymbol{A}_{1/r,0}(S_{j-1} \cap S_j, \mathbb{E}), \quad 1 \leq j \leq r, \\ &f_r(z) = f_0(z e^{-2\pi i}), \quad z \in S_r. \end{aligned}$$

4. Conclude $f_j \in \boldsymbol{A}_{1/r}(S_j, \mathbb{E})$, $0 \leq j \leq r$.

8
Solutions of Highest Level

In this chapter we are going to prove that the formal transformations occurring in the *Splitting Lemma* (p. 42) and in Theorem 11 (p. 52) are, in fact, r-summable in the sense of Chapter 6. Based upon this, we shall then show that highest-level formal fundamental solutions are k-summable for $1/k = s$ equal to their Gevrey order. As an application of this result we then prove the factorization of formal fundamental solutions according to *Ramis* and the author, as described in Chapter 14. For every highest-level formal fundamental solution (HLFFS for short), we then define *normal solutions of highest level* that were first introduced by the author in [8], and we shall investigate their properties in more detail, defining corresponding Stokes' directions, Stokes' multipliers, etc.

Since in Chapter 3 all power series have been in the variable $1/z$, we have to make some more or less obvious adjustments in the definition of r-summability. In particular, we define *sectorial regions at infinity* to be such regions G that by the inversion $z \mapsto 1/z$ are mapped onto sectorial regions in the previous sense, and from now on, we shall denote by $\hat{f}(z)$ *a formal power series in the variable* z^{-1}. It is then natural to say that

- a power series $\hat{f}(z) \in \mathbb{E}\,[[z^{-1}]]$ is k-summable in direction d if and only if $\hat{f}(z^{-1}) = \sum_0^\infty f_n\, z^n$ is k-summable in direction $-d$.

The formal Borel transform $\hat{\mathcal{B}}_k \hat{f}$ is defined by applying $\hat{\mathcal{B}}_k$ to $\hat{f}(z^{-1})$, and its sum then is holomorphic in a small sector of bisecting direction $-d$, and is of exponential growth at most k there. The k-sum of $\hat{f}(z)$ then is obtained by summing $\hat{f}(z^{-1})$ followed by the change of variable $z \mapsto 1/z$. Hence, *if*

$f(z)$ is the k-sum in direction d of a power series $\hat{f}(z)$ in inverse powers of z, then $f(z)$ is holomorphic in a sectorial region $G(d, \alpha)$ (near infinity) of bisecting direction d and opening $\alpha > \pi/k$, and $f(z)$ can be represented by a Laplace integral (5.1), with z replaced by $1/z$ and the direction of integration τ close to $-d$. So here, turning the path of integration in the positive sense leads to holomorphic continuation of $f(z)$ *in the negative sense*, and vice versa.

Because HLFFS in general are series in some root of z^{-1}, we need to generalize the notion of k-summability to q-meromorphic transformations, or more generally to arbitrary formal Laurent series in the variable $z^{-1/q}$:

- A power series in $z^{-1/q}$ will be called k-summable in direction d, if the series obtained by the change of variable $z = w^q$ is qk-summable in direction d/q. Compare Exercise 2 on p. 72 to see that this definition gives good sense even if the series we start with accidentally happens to not contain any roots.

- A *formal Laurent series* $\sum_{n=-m}^{\infty} f_n\, z^{-n}$ is called k-summable in direction d if and only if its power series part $\hat{f}(z) = \sum_{n=0}^{\infty} f_n\, z^{-n}$ is summable in this sense, with the k-sum of the Laurent series equal to the k-sum of the power series part plus the finite principle part $\sum_{n=-m}^{-1} f_n\, z^{-n}$. Compare this to Exercise 5 on p. 72.

8.1 The Improved Splitting Lemma

In the construction of HLFFS we had to consider formal analytic transformations in two places: In the *Splitting Lemma* in Section 3.2, and later in Section 3.4, finding a transformation so that the new system is rational. To proceed, we first reconsider the proof of the Splitting Lemma and show that, beginning with a convergent system (3.1), the transformation as well as the transformed system are r-summable, and we can even completely describe the set of singular directions.

Lemma 11 (Improved Splitting Lemma) *Consider a convergent system* (3.1) (p. 37) *satisfying the assumptions of the Splitting Lemma on p. 42. Then the formal matrix power series $\hat{T}_{12}(z)$ and the block $\hat{B}_{22}(z)$ of the coefficient matrix of the transformed system are r-summable in every direction d except for the finitely many singular ones of the form $-rd = \arg(\mu_1 - \mu_2)$ mod 2π, with μ_j an eigenvalue of $A_0^{(jj)}$. The same statement, but with μ_1, μ_2 interchanged, holds for $\hat{T}_{21}(z)$ and $\hat{B}_{11}(z)$.*

Proof: Analogously to the proof of Theorem 11 (p. 52), we define $T(u) = \sum_1^\infty T_{12}\, u^{n-r}/\Gamma(n/r)$, $B(u) = \sum_1^\infty B_{22}\, u^{n-r}/\Gamma(n/r)$, and (slightly abusing notation) $A_{jk}(u) = \sum_1^\infty A_n^{(jk)}\, u^{n-r}/\Gamma(n/r)$. Then $u^{r-1}\, A_{jk}(u)$ are entire functions of exponential growth at most r, owing to the convergence of (3.1). Moreover, (3.6) and (3.7) (p. 42) are formally equivalent to the integral equation

$$\begin{aligned} T(u)\, A_0^{(22)} - (A_0^{(11)} + r u^r I)\, T(u) \quad &= \\ A_{12}(u) + \int_0^u \Big[A_{11}([u^r - t^r]^{1/r})\, T(t) - T(t)\, B([u^r - t^r]^{1/r})\Big]\, dt^r, & \quad (8.1) \end{aligned}$$

with

$$B(u) = A_{22}(u) + \int_0^u A_{21}([u^r - t^r]^{1/r})\, T(t)\, dt^r. \qquad (8.2)$$

The Splitting Lemma ensures existence of $T(u)$ and $B(u)$, holomorphic and single-valued near the origin and satisfying (8.1). We aim at proving that both can be holomorphically continued into the largest star-shaped[1] region G that does not contain any point u for which $A_0^{(22)}$ and $A_0^{(11)} + r u^r I$ have an eigenvalue in common. Observe Exercise 3 on p. 214 to see that this is the largest star-shaped set not containing any solution of the equation $r u^r = \mu_2 - \mu_1$. Moreover, we shall show that in this region both matrices have exponential growth at most r. This then implies r-summability of $\hat{T}_{12}(z)$ and $\hat{B}_{22}(z)$, with singular directions as stated, and one can argue analogously for the other two blocks.

To do all this, we employ an iteration: Beginning with $B(u; 0) \equiv 0$, $T(u; 0) \equiv 0$, we plug $B(u; m)$ and $T(u; m)$ into the right-hand sides of (8.1), (8.2) and determine $T(u; m+1)$ from the left-hand side of (8.1), resp. let $B(u; m+1)$ be equal to $B(u)$ in (8.2). Both sequences so obtained are holomorphic in G, except for a pole of order at most $r - 1$ at the origin. For the following estimates, we choose d so that the ray $u = x e^{-id}$, $x \geq 0$, is in G, hence d is a non-singular direction. Then $\|A_{jk}(u)\| \leq \sum_1^\infty a^n\, x^{n-r}/\Gamma(n/r)$, with $a \geq 0$ *independent of* d. Inductively we show estimates of the form $\|T(u; m)\| \leq \sum_1^\infty t_n^{(m)}\, x^{n-r}/\Gamma(n/r)$, $\|B(u; m)\| \leq \sum_1^\infty b_n^{(m)}\, x^{n-r}/\Gamma(n/r)$: For $m = 0$ this is certainly correct with $t_n^{(0)} = b_n^{(0)} = 0$. Given this estimate for some $m \geq 0$, we use the recursion formulas and the Beta Integral (p. 229) to show the same type of estimate for $m + 1$. In particular, we may set

$$\left.\begin{aligned} b_n^{(m+1)} \quad &= \quad a^n + \textstyle\sum_1^{n-1} a^{n-k} t_k^{(m)} \\ t_n^{(m+1)} \quad &= \quad c\left[a^n + \textstyle\sum_1^{n-1} (a^{n-k} + b_{n-k}^{(m)}) t_k^{(m)}\right] \end{aligned}\right\} \quad n \geq 1,$$

where c is a constant which arises when solving the left-hand side of (8.1) for $T(u)$ and then estimating the solution – hence c is independent of n and m, as well as independent of d provided we alter d only slightly so that

[1] With respect to the origin.

it keeps a positive distance from the singular directions. For every n, the numbers $t_n^{(m)}$, $b_n^{(m)}$ can be seen to be monotonically increasing with respect to m and become constant when $m \geq n$. Their limiting values t_n, b_n then satisfy the same recursion equations, with the superscripts dropped. Setting $b(x) = \sum_1^\infty b_n\, x^n$, $t(x) = \sum_1^\infty t_n\, x^n$, $a(x) = \sum_1^\infty a^n\, x^n = ax(1-ax)^{-1}$, we obtain formally $b(x) = a(x)[1+t(x)]$, $t(x) = c[a(x)+(a(x)+b(x))t(x)]$. This can be turned into a quadratic equation for $t(x)$, having one solution $t_1(x)$ that is a holomorphic function near the origin, while the other one has a pole there. The coefficients of $t_1(x)$ satisfy the same recursion formulas as t_n, *so are in fact equal to* t_n. This implies that both t_n and b_n cannot grow faster than some constant to the power n. From this fact we conclude that $T(u;m)$ and $B(u;m)$ can be estimated by $c\,x^{1-r} e^{Kx^r}$ with suitably large c, K *independent of* m. Therefore, the proof will be completed provided that we show convergence of $T(u;m)$ and $B(u;m)$ as $m \to \infty$. This can be done by deriving estimates, similar to the ones above, for the differences $T(u;m) - T(u;m-1)$ and $B(u;m) - B(u;m-1)$ and turning the sequence into a telescoping sum. For details, compare the exercises below. □

It is important to note for later applications that the above lemma not only ensures r-summability of the transformation $\hat{T}(z)$, *but also allows explicit computation of the possible singular directions in terms of parameters of the system* (3.1) (p. 37). As we shall make clear in Chapter 9, it may happen that some, or all, of these directions are nonsingular, depending on whether some entries in Stokes' multipliers are zero. However, in generic cases *all these directions are singular!*

Exercises: In the following exercises, make the same assumptions as in the lemma above, and use the same notation as in its proof.

1. For $m \geq 1$ and u as in the proof of the above lemma, show

$$\begin{aligned} \|T(u;m) - T(u;m-1)\| &\leq \sum_{n=m}^{\infty} d_n^{(m)}\, x^{n-r}/\Gamma(n/r), \\ \|B(u;m) - B(u;m-1)\| &\leq \sum_{n=m}^{\infty} \tilde{d}_n^{(m)}\, x^{n-r}/\Gamma(n/r), \end{aligned}$$

   with constants $d_n^{(m)}$, $\tilde{d}_n^{(m)}$ satisfying

$$d_n^{(m+1)} = \sum_{k=m}^{n-1} b^{n-k}(d_k^{(m)} + \tilde{d}_k^{(m)}), \quad \tilde{d}_n^{(m+1)} = \sum_{k=m}^{n-1} a^{n-k} d_k^{(m)},$$

   for suitably large $b > 0$, and a as in the proof of the lemma.

2. Setting $d_m(x) = \sum_{n=m}^\infty d_n^{(m)} x^n$, and analogously for $\tilde{d}_m(x)$, derive a recursion for these functions – in particular, conclude convergence of the series for $0 < x < \min(a, b)$.

3. Setting $d(x) = \sum_m d_m(x)$, $\tilde{d}(x) = \sum_m \tilde{d}_m(x)$, use the previous exercise to show functional equations, and from these compute the functions in terms of d_1, $\tilde{d}_1$. In particular, show that both functions are holomorphic near the origin. Use this to conclude existence of d so that $d_n^{(m)}, \tilde{d}_n^{(m)} \le d^n$ for every n, m.

4. Use the previous exercises to show that $T(u; m)$ and $B(u; m)$ converge uniformly on every compact subset of the region G defined in the proof of the above lemma.

8.2 More on Transformation to Rational Form

In Theorem 11 (p. 52) we showed that a formal system can be transformed into a rational one – meaning a system with a coefficient matrix being a rational function with poles at the origin and infinity only. Here we shall prove that, starting with a system (3.3) whose coefficient matrix is r-summable, then so is the transformation obtained in the said theorem. Indeed, much more can be said:

Proposition 19 *For some $d \in \mathbb{R}$, let $zx' = \hat{A}(z)\,x$ and $z\tilde{x}' = \hat{B}(z)\,\tilde{x}$ be two formal system of Poincaré rank $r \ge 1$ whose coefficient matrices are r-summable in some direction d, and let $\hat{T}(z)$ be a formal meromorphic transformation of Gevrey order $s = 1/r$ satisfying* (3.5)*. Furthermore, assume that $-rd \ne \arg(\mu - \tilde{\mu}) \bmod 2\pi$, for any two distinct eigenvalues μ, $\tilde{\mu}$ of the leading term A_0. Then $\hat{T}(z)$ is also r-summable in direction d.*

Proof: Using Exercise 4 on p. 41, applied to the transpose of $\hat{T}(z)$, we factor $\hat{T}(z) = \hat{T}_0(z)\ T(z)$ with a terminating meromorphic transformation $T(z)$ and a formal analytic transformation $\hat{T}_0(z)$ of Gevrey order s and leading term I. Setting $\hat{B}_0(z) = (T(z)\hat{B}(z) + zT'(z))T^{-1}(z)$, we see that $\hat{B}_0(z)$ is also r-summable in direction d, while $\hat{T}(z)$ is r-summable in direction d if and only if the same holds for $\hat{T}_0(z)$. This shows that in the proof we may without loss of generality assume that $\hat{T}(z)$ is a formal analytic transformation with constant term I, and then the leading term of $\hat{B}(z)$ is equal to A_0, and its Poincaré rank is again equal to r.

Using the same notation as in the proof of Theorem 11 (p. 52), we conclude from our assumptions that the matrices $u^{r-1}\,A(u)$ and $u^{r-1}\,B(u)$ are holomorphic in $G = D(0, \rho) \cup S(-d, \varepsilon)$, for sufficiently small $\rho, \varepsilon > 0$, and of exponential growth at most r there. Moreover, we conclude that $T(u)$ is holomorphic in $D(0, \rho)$, with ρ small enough, except for a pole of order at most $r - 1$, and satisfies the integral equation (3.17) (p. 53) for $|u| < \rho$. Making the above $\varepsilon > 0$ smaller if necessary, we can arrange that the equation $ru^r = \mu - \tilde{\mu}$, with $\mu, \tilde{\mu}$ as above, does not have a solution in

G. Then this integral equation, according to the exercises below, implies that $T(u)$ can be holomorphically continued into G and is of exponential growth at most k there. This, however, is equivalent to k-summability of $\hat{T}(z)$ in direction d. □

It is worthwhile emphasizing that the integral equation (3.17) (p. 53), under the assumptions stated above, in general is *singular at the origin*, meaning roughly that one cannot use the standard iteration procedure to show existence of a solution near the origin. However, *once a solution is known, the integral equation can be used for holomorphic continuation, and a growth estimate for the solution follows from analogous estimates of the terms in the equation.*

In the construction of HLFFS, Theorem 11 (p. 52) is applied to the diagonal blocks of a system obtained by an application of the Splitting Lemma. These blocks have been shown above to be r-summable, and by construction their leading term only has one eigenvalue. In this situation, the above proposition shows that *no additional singular rays occur*, because the condition in terms of the eigenvalues of A_0 is void.

Exercises: In the following exercises, use the same notation and assumptions as in the proof of the above proposition.

1. For fixed $u_0 = x_0 e^{i\tau}$ with $|d + \tau| < \varepsilon$, assume that $T(u)$ has been holomorphically continued along the line segment $u = x e^{i\tau}$, $0 \le x \le x_0$, which trivially holds for $x_0 < \rho$. Define $C(u) = A(u) - B(u) + \int_0^{u_0} [\, A([u^r - t^r]^{1/r})\, T(t) - T(t)\, B([u^r - t^r]^{1/r})\,]\, dt^r$, and show:

   (a) The function $C(u)$ is holomorphic in a small region $G_{u_0,\delta} = \{u : |\tau - \arg(u - u_0)| < \delta\}$, $\delta > 0$.

   (b) The integral equation $T(u)\, A_0 - (A_0 + r u^r I)\, T(u) = C(u) + \int_{u_0}^{u} [\, A([u^r - t^r]^{1/r})\, T(t) - T(t)\, B([u^r - t^r]^{1/r})\,]\, dt^r$ has a unique solution $T(u)$ that is holomorphic in $G_{u_0,\delta}$ and of exponential growth at most r.

2. Show that the function $T(u)$ is holomorphic in G and of exponential growth not more than r there.

3. Prove the following **Corollary to Proposition 19:** In addition to the assumptions of Proposition 19, let $\hat{A}(z)$ and $\hat{B}(z)$ be convergent, and let A_0 have only one eigenvalue. Then the transformation $\hat{T}(z)$ converges as well.

8.3 Summability of Highest-Level Formal Solutions

On p. 55 we have given the definition of HLFFS and its data pairs. Using the results of the previous sections, we are now ready to investigate summability of HLFFS. As will become clear later, the following theorem, in a way, is the main one in our theory of HLFFS:

Theorem 42 (Main Theorem) *Assume that we are given a system* (3.1) (p. 37) *having an essentially irregular singularity at infinity. Then the following holds true:*

(a) *For any two HLFFS of* (3.1)*, the data* p, q *agree. In particular, both HLFFS are of the same Gevrey order* $s = 1/k$*,* $k = r - p/q$*.*

(b) *For any two HLFFS of* (3.1)*, the data pairs* $(q_1(z), s_1), \ldots, (q_\mu(z), s_\mu)$ *agree modulo renumeration, so in particular the number* μ *of such pairs is the same.*

(c) *For any two HLFFS* $\hat{F}_1(z)$*,* $\hat{F}_2(z)$*, assume that their data pairs coincide, which can always be brought about by suitably permuting the columns of any one of the HLFFS. Then the transformation* $T(z) = \hat{F}_1^{-1}(z)\,\hat{F}_2(z)$ *is diagonally blocked of type* $(s_1, \ldots, s_\mu)$ *and converges.*

(d) *Every HLFFS is* k*-summable in all directions* d*, except for*

$$-kd = \arg(\lambda_j^{(p)} - \lambda_m^{(p)}) \mod 2\pi, \quad j \neq m,\ 1 \leq j, m \leq \mu. \tag{8.3}$$

Proof: Let $\hat{F}_j(z)$, $1 \leq j \leq 2$, be any two HLFFS of (3.1), and let $B_j(z)$ be the coefficient matrices of the corresponding transformed systems, as defined on p. 55. Setting $\hat{T}(z) = \hat{F}_1^{-1}(z)\,\hat{F}_2(z)$, one can show formally

$$z\,\hat{T}'(z) = B_1(z)\,\hat{T}(z) - \hat{T}(z)\,B_2(z). \tag{8.4}$$

Let $q = m_1\,q_1 = m_2\,q_2$ be the least common multiple of q_1, q_2. Originally, the matrix $B_j(z)$ has been expanded with respect to the variable $w_j = z^{1/q_j}$, and the first p_j coefficients are scalar multiples of the unit matrix. Re-expanding $B_j(z)$ in the new variable $w = z^{1/q}$, the first $m_j p_j$ coefficients now are scalar, and some may even be zero. Assume for the moment that both p_j are positive, so that scalar coefficients occur. Make the change of variable $z = w^q$ in (8.4), and then apply Exercise 2, with $\mu = \tilde{\mu} = 1$, i.e., $\hat{A}(z) = B_1(z^q)$, $\hat{B}(z) = B_2(z^q)$ blocked trivially. This implies that the highest terms of both $B_j(z)$ are *the same scalar multiple of the identity matrix*. Since these terms commute with $\hat{T}(z)$, they can be cancelled from (8.4). Then, the same argument can be used over again, until one of the highest terms, after cancellation of the ones shown to be identical, is no longer scalar. In this situation, an application of Exercise 2 implies that then *both leading terms must have more than one eigenvalue*; hence $m_1 p_1 = m_2 p_2$ follows. Together with $q = m_1 q_1 = m_2 q_2$, this implies $p_1/q_1 = p_2/q_2$. By assumption p_j, q_j are co-prime except for $q_j = 1$; so we conclude $q_1 = q_2$ and $p_1 = p_2$. This shows (a). However, applying Exercise 2 to the situation where both leading terms have several eigenvalues, we even obtain statements (b) and (c).

To show (d), observe that according to (c), it suffices to show the existence of *one* HLFFS that is summable as stated. This, however, is a consequence of Lemma 11 (p. 124) and Proposition 19 (p. 127), combined with the algorithm of constructing an HLFFS as stated on p. 50. □

The above Main Theorem not only implies k-summability of HLFFS. It also supplies information on *the degree of freedom* in these objects:

On one hand, it is shown that the data pairs, up to enumeration, *correspond uniquely to a given system* (3.1), and to every enumeration of the data pairs one can find an HLFFS, simply by a suitable permutation of its columns. Moreover, the q-meromorphic transformation $\hat{F}(z)$ is defined *up to a left-hand-side factor* $T(z)$ that is a diagonally blocked convergent q-meromorphic transformation. This transformation, owing to the definition of HLFFS, transforms one system $z\,y' = B(z)\,y$ into another one, say, $z\,\tilde{y}' = \tilde{B}(z)\,\tilde{y}$, and both coefficient matrices have terminating expansions in the variable $w = z^{1/q}$. It is clear that, given one HLFFS, multiplication from the left with such a factor $T(z)$ again gives an HLFFS. Unfortunately, it is not at all obvious which diagonally blocked q-meromorphic transformations, when used to transform a system $z\,y' = B(z)\,y$ with terminating expansion, will produce another one having the same property. This, however, will not be of importance for us.

Exercises:

1. Let $A_n \in \mathbb{C}^{\nu\times\nu}$, $B_n \in \mathbb{C}^{\tilde{\nu}\times\tilde{\nu}}$, with $r \ge 0$, $\nu, \tilde{\nu} \ge 1$ not necessarily equal, and $n \ge 0$. Let $T_n \in \mathbb{C}^{\nu\times\tilde{\nu}}$, $n \ge n_0$ ($n_0 \in \mathbb{Z}$) satisfy

$$-(n-r)\,T_{n-r} = \sum_{m=n_0}^{n} \left(A_{n-m}\,T_n - T_n\,B_{n-m}\right), \quad n \ge n_0,$$

with $T_n = 0$ for $n < n_0$. If $r \ge 1$ and A_0, B_0 have disjoint spectrum, show that $T_n = 0$ for every $n \ge n_0$ follows. For $r = 0$, find a necessary and sufficient condition in terms of the spectra of A_0, B_0 under which the same conclusion holds.

2. Let a formal meromorphic transformation $\hat{T}(z)$ satisfy (3.5) (p. 40), with matrices $\hat{A}(z)$, $\hat{B}(z)$ of the following form:

   (a) The matrices $z^{-r}\hat{A}(z)$ and $z^{-r}\hat{B}(z)$ are formal matrix power series in z^{-1} with constant terms A_0 resp. B_0 that may vanish or not (in other words: the corresponding formal systems have Poincaré ranks at most r), and both A_0 and B_0 are in Jordan form, say,

$$\begin{aligned} A_0 &= \operatorname{diag}[\lambda_1 I_{s_1} + N_1, \ldots, \lambda_\mu I_{s_\mu} + N_\mu], \\ B_0 &= \operatorname{diag}[\tilde\lambda_1 I_{\tilde s_1} + \tilde N_1, \ldots, \tilde\lambda_{\tilde\mu} I_{\tilde s_{\tilde\mu}} + \tilde N_{\tilde\mu}], \end{aligned}$$

with distinct $\lambda_1, \ldots, \lambda_\mu$, resp. $\tilde\lambda_1, \ldots, \tilde\lambda_{\tilde\mu}$ and nilpotent matrices N_j, resp. $\tilde N_j$.

(b) The matrices $\hat A(z)$ and $\hat B(z)$ are diagonally blocked in the block structures of their leading terms A_0, resp. B_0.

Show that then $\tilde\mu = \mu$, and the pairs (λ_j, s_j) and $(\tilde\lambda_j, \tilde s_j)$ agree up to a renumeration. In case the pairs are identical, show that $\hat T(z)$ is diagonally blocked, and converges if both $\hat A(z)$ and $\hat B(z)$ converge.

3. Under the assumptions of the previous exercise, assume that the pairs (λ_j, s_j) and $(\tilde\lambda_j, \tilde s_j)$ are *not* identical. Show that then one can permute the columns, resp. the rows, of $\hat T(z)$ so that the resulting matrix is diagonally blocked in the block structure of $\hat A(z)$, resp. of $\hat B(z)$.

8.4 Factorization of Formal Fundamental Solutions

We now are going to discuss briefly the classical notion of *formal fundamental solutions* (FFS for short) of a system (3.1) (p. 37). Such a FFS of a convergent system will, in general, not be k-summable for any $k > 0$. However, we are going to show that it can always be factored into finitely many terms that individually are k-summable, but for values k depending on the factor.

In principle, one can compute a *formal fundamental solution* even for formal systems (3.3) (p. 40). To do so is equivalent to finding a q-meromorphic formal transformation $\hat F(z)$ producing an *elementary formal transformed system*, say, with coefficient matrix $\hat B(z) = w^{\tilde r} \sum_0^\infty B_n w^{-n}$, $w = z^{1/q}$, where all B_n commute with one another. It follows from the exercises below that then another q-meromorphic transformation takes (3.3) into

$$z\, y' = B(z)\, y, \quad B(z) = \operatorname{diag}[b_1(w) I_{s_1} + N_1, \ldots, b_\mu(w) I_{s_\mu} + N_\mu], \tag{8.5}$$

with $w = z^{1/q}$ and $B(z)$ *in normalized Jordan canonical form*, meaning that

- the $b_m(w)$ are distinct polynomials of degree at most $\tilde r$, with constant terms having a real part in the half-open interval $[0, 1/q)$, and

- the N_m are nilpotent Jordan matrices whose blocks are ordered so that they increase in size.

Therefore, we shall always *assume that for a formal fundamental solution* $\hat{F}(z)$ *the transformed system already is of the form* (8.5). The system (8.5) then has a unique fundamental solution of the form

$$Y(z) = \operatorname{diag}\left[e^{q_1(z)} z^{J_1}, \ldots, e^{q_\mu(z)} z^{J_\mu}\right],$$

with $z\, q_m'(z) = b_m(w) - b_m(0)$, and $J_m = b_m(0) I_{s_m} + N_m$. So each q_m is a polynomial in the qth root of z without constant term, and J_m a constant matrix in Jordan form, with a single eigenvalue whose real part is in $[0, 1/q)$. In analogy to the phrase used for HLFFS, we call the pairs $(q_j(z), J_j)$, $1 \le j \le \mu$, *the data pairs of the FFS.* Recall that the $b_m(w)$ were all distinct to see the same for the data pairs. Moreover, note that the enumeration of the data pairs is always chosen to correspond to the enumeration of the $b_m(w)$ in (8.5). In particular, a renumeration of the data pairs reflects in a suitable permutation of the columns of the FFS. As for HLFFS, the data pairs of a FFS will be shown to *be closed with respect to continuation*, meaning that to every pair $(q_m(z), J_m)$ there exists a pair $(q_\kappa(z), J_\kappa)$ with $q_\kappa(z) = q_m(z e^{2\pi i})$ and $J_m = J_\kappa$. This κ even is uniquely determined by m, because the data pairs are all distinct!

We now show existence of FFS. For simplicity, we restrict ourselves to convergent systems (3.1); however, one can check that the same statements remain correct for formal systems (3.3).

Proposition 20 *Every system* (3.1) (p. 37) *has a FFS of the form described above. Moreover, the following always holds:*

(a) *For any two FFS of* (3.1), *the data pairs coincide up to renumeration.*

(b) *The data pairs of any FFS are closed with respect to continuation.*

(c) *For any two FFS* $\hat{F}_1(z)$, $\hat{F}_2(z)$, *assume that their data pairs coincide, which can always be brought about by suitably permuting the columns of any one of the FFS. Then* $\hat{F}_1^{-1}(z)\, \hat{F}_2(z)$ *is diagonally blocked of type* $(s_1, \ldots, s_\mu)$ *and constant.*

Proof: To show existence of FFS, first assume that infinity is an almost regular-singular point. Then by definition, a terminating meromorphic transformation $T(z)$ and a polynomial $q(z)$ of degree at most r exist, so that the combined transformation $x = e^{q(z)}\, T(z)\, \tilde{x}$ leads to a system with a regular-singular point at infinity. According to the results of Section 2, in particular Theorem 6 (p. 32),[2] there exists another meromorphic transformation $\tilde{T}(z)$, transforming this new system into one with constant

[2]Use a change of variable $z = 1/w$ to transfer the singularity to the origin.

coefficient matrix $B(z) \equiv B_0$, having eigenvalues with real parts in $[0,1)$. Finally, a constant transformation may be used to put B_0 into Jordan canonical form $J = \operatorname{diag}[J_1, \ldots, J_\mu]$, where each J_m contains all blocks corresponding to one eigenvalue, those blocks being ordered according to size. Since the scalar exponential shift $e^{q(z)}$ commutes with every other transformation, this altogether shows existence of a meromorphic transformation $F(z)$, transforming (3.1) into $z\,y' = B(z)\,y$, with $B(z) = z\,q'(z) + J$. It follows from the definition that $F(z)$ is a FFS, with data pairs $(q(z), J_m)$.

Observe that, according to Exercise 1 on p. 58, for dimension $\nu = 1$ the system (3.1) always is almost regular-singular. Hence we may now proceed by induction with respect to ν, and we may assume that infinity is an essentially irregular singularity. Let then $\hat{F}_1(z)$ be a q-meromorphic HLFFS of (3.1). This transformation takes (3.1) into a system with diagonally blocked coefficient matrix $B(z)$ having diagonal blocks $B_m(z)$ of strictly smaller dimension. By induction hypothesis, the systems with coefficient matrices $q^{-1}\,B_m(w^q)$ have FFS; hence we obtain a FFS $\hat{F}_2(w)$ for the transformed system as their direct sum. Setting $\hat{F}(z) = \hat{F}_1(z)\,\hat{F}_2(z^{1/q})$, it is not difficult to verify from the definition that this is a FFS of (3.1), with data pairs obtained from those of $\hat{F}_2(w)$ by change of variable $w = z^{1/q}$.

To show (a)–(c), conclude from the definition that for any FFS $\hat{F}(z)$, the matrix $\hat{F}(z e^{2\pi i})$ is again a FFS whose data pairs are obtained from those of the first one by continuation. Hence (b) follows from (a). Now, let $\hat{F}_1(z)$, $\hat{F}_2(z)$ be two FFS, and let $\hat{T}(z) = \hat{F}_1^{-1}(z)\,\hat{F}_2(z)$. Observe that we may assume without loss in generality that both $\hat{F}_j(z)$ are q-meromorphic, with q independent of j, and then set $z = w^q$. Corresponding to each FFS $\hat{F}_j(z)$, let

$$B_j(z) = \operatorname{diag}[b_1^{(j)}(w) I_{s_1^{(j)}} + N_1^{(j)}, \ldots, b_\mu^{(j)}(w) I_{s_\mu^{(j)}} + N_\mu^{(j)}]$$

be as in (8.5). We then have that $\hat{T}(z)$ formally satisfies (8.4). Blocking rows, resp. colums, according to the block structure of $B_1(z)$, resp. $B_2(z)$, we obtain $\hat{T}(z) = [\hat{T}_{\kappa m}(z)]$, with *not necessarily quadratic diagonal blocks*. From (8.4) we conclude, using Exercise 1 on p. 130, that $T_{\kappa m}(z) = 0$ except when $b_\kappa^{(1)}(w) \equiv b_m^{(2)}(w)$. Since $\hat{T}(z)$ is invertible, we conclude that to every m there exists a $\kappa = \sigma(m)$ so that this holds, and this κ even is unique, since all $b_\kappa^{(1)}(w)$ are distinct. Hence, after a suitable permutation of the columns of either one of the $\hat{F}_j(z)$, we may assume without loss of generality that $\sigma(m) = m$, i.e., $\hat{T}(z)$ is diagonally blocked. Using invertibility of $\hat{T}(z)$ again, we then find $s_m^{(1)} = s_m^{(2)}$, for every m. Moreover, the square diagonal blocks satisfy $q\,w(d/dw)\,\hat{T}_{mm}(w^q) = N_m^{(1)}\,\hat{T}_{mm}(w^q) - \hat{T}_{mm}(w^q)\,N_m^{(2)}$. From Exercise 6 we conclude that $\hat{T}_{mm}(z)$ is constant, so $N_m^{(1)}$ and $N_m^{(2)}$ are similar. Owing to our assumption that their blocks are ordered according to size, they must in fact be equal. This observation completes the proof of statements (a) and (c). □

The above result shows the existence, and describes the degree of freedom, of FFS. To describe their summability properties, it is necessary to introduce some notation: For an arbitrary nonzero polynomial $p(z)$ in some root of z, let $\deg p$ stand for the rational exponent of the highest power of z occurring in $p(z)$. Given the data pairs of a FFS, we introduce the rational numbers

$$d_{jm} = \deg(p_j - p_m), \quad \text{if } p_j(z) \not\equiv p_m(z),\ 1 \le j, m \le \mu.$$

They form a finite, possibly empty, set of rational numbers, which are all positive, since by definition of the p_j their constant terms all vanish. We write this set in the form $\{k_1 > k_2 > \ldots > k_\ell > 0\}$, referring to it as *the level set of* the FFS $\hat{F}(z)$. The integer $\ell \ge 0$ will be referred to as *the number of levels of* $\hat{F}(z)$. In case $\ell \ge 1$, we shall call k_1 *the highest level* of $\hat{F}(z)$. From now on, consider data pairs with $\ell \ge 1$. For $1 \le \mu \le \ell$, a number $d \in \mathbb{R}$ will be called *singular of level* μ, provided that polynomials p_j, p_m exist for which

$$p_j(z) - p_m(z) = p_{jm}\, z^{k_\mu} + \text{lower powers}, \quad \arg p_{jm} = -d\, k_\mu. \tag{8.6}$$

In other words: $\exp[p_j(z) - p_m(z)]$ has *degree* k_μ *and maximal growth rate along the ray* $\arg z = d$. Furthermore, we write $j \sim m$ of level μ, if $\deg(p_j - p_m) < k_\mu$. This way, we obtain a number of equivalence relations that become finer with increasing μ, in the sense that fewer numbers m are equivalent. We now say that the data pairs $(q_m(z), J_m)$ are *ordered* provided that polynomials with indices that are equivalent of level μ come consecutively, for every $\mu = 1, \ldots, \ell$. Note that we can always rearrange the columns of the FFS $\hat{F}(z)$ and simultanuously permute its data pairs, so that first polynomials whose indices are equivalent of level $\mu = 1$ come consecutively, then further permute pairs within each equivalence class of level 1 to make polynomials come together whenever their indices are equivalent of level 2, and so on. In other words, *every system* (3.1) *has a FFS with ordered data pairs.*

Consider now an ordered set of data pairs. Then to each equivalence relation of level μ, there corresponds a block structure of arbitrary $\nu \times \nu$ matrices, by grouping their rows and columns together once their indices are equivalent of that level. This block structure will be referred to as the *μth block structure*, or *the block structure of level* μ. In particular, for $\mu = 1$ we speak of the *block structure of highest level.* For convenience, we also introduce the *trivial block structure of level zero*, where we do not subdivide matrices at all. These block structures were studied in [34, 35] under the name *iterated block structure.*

Using this terminology, we can now formulate the following basic result on FFS:

Theorem 43 *For an arbitrary FFS* $\hat{F}(z)$ *of a system* (3.1) (p. 37), *the following holds:*

(a) *The FFS has empty level set if and only if infinity is an almost regular-singular point of* (3.1)*, and in this case* $\hat{F}(z)$ *converges.*

(b) *If the number* ℓ *of its levels is positive, then its highest level is not larger than the Poincaré rank* r *of* (3.1)*. Moreover,* $k_1 = r$ *holds if and only if the leading term of* (3.1) *has several eigenvalues.*

(c) *Let infinity be an essentially irregular singularity of* (3.1)*, so that* $\ell \geq 1$*. Then* $\hat{F}(z)$ *can be factored as*

$$\hat{F}(z) = \hat{F}_1(z) \cdot \ldots \cdot \hat{F}_\ell(z),$$

where each $\hat{F}_\mu(z)$ *is a* k_μ*-summable* q*-meromorphic formal transformation, whose singular directions are among those given by* (8.6)*. In particular,* $\hat{F}(z)$ *is of Gevrey order not larger than* $s = 1/k_\ell$*, and* $\hat{F}_1(z)$ *is an HLFFS of* (3.1)*.*

(d) *In case of* $\ell \geq 2$*, let* $\hat{G}_1(z) \cdot \ldots \cdot \hat{G}_\ell(z)$ *be another factorization of* $\hat{F}(z)$ *as described in* (c)*. Then there exist convergent meromorphic transformations* $T_2(z), \ldots, T_\ell(z)$ *so that with* $T_1(z) = T_{\ell+1}(z) = I$ *we have*

$$\hat{F}_m(z)\, T_{m+1}(z) = T_m(z)\, \hat{G}_m(z), \quad 1 \leq m \leq \ell. \tag{8.7}$$

(e) *If the data pairs of* $\hat{F}(z)$ *are ordered, then one can find a factorization as above, for which each factor* $\hat{F}_j(z)$ *is diagonally blocked in the block structure of level* $j-1$*.*

Proof: First, observe that if the theorem holds for *some* FFS, then it is correct for *every other* FFS, owing to parts (a) and (c) of Proposition 20. So it suffices to construct a particular FFS. To do so, follow the same steps, and use the same notation, as in the proof of Proposition 20. If infinity is an almost regular-singular point, one can read off from there existence of a convergent FFS with empty level set. In the other case, an FFS exists having the form $\hat{F}(z) = \hat{F}_1(z)\, \hat{F}_2(z^{1/q})$, with an HLFFS $\hat{F}_1(z)$, while $\hat{F}_2(w)$ is the direct sum of at least two blocks. Each of these blocks is an FFS of strictly smaller systems, and by induction hypothesis we can assume that the theorem holds for those. In particular, we may assume that the data pairs of the blocks are ordered and a factorization as in (e) exists. Making the change of variable $w = z^{1/q}$, and then combining the data pairs of the diagonal blocks in their natural order, gives the data pairs for the FFS. From the definition of, resp. the algorithm to construct, an HLFFS, one can read off that the leading terms of these data pairs coincide with what was introduced on p. 56 as the data pairs of the HLFFS $\hat{F}_1(z)$. This implies that the data pairs of $\hat{F}(z)$ are ordered, and their highest-level k_1 coincides with the number $1/s$ defined there. From Theorem 42 we then conclude that $\hat{F}_1(z)$ is k_1-summable, with singular directions that are among those introduced above as being singular of level 1. This, together with the induction hypothesis, completes the proof of (a)–(c), and (e). To show (d), define $T_m(z)$ by (8.7), and note that then

$$\hat{F}_m(z)\, T_{m+1}(z)\, \hat{G}_m^{-1}(z) = \hat{F}_{m-1}^{-1}(z) \cdot \ldots \cdot \hat{F}_1^{-1}(z)\, \hat{G}_1(z) \cdot \ldots \cdot \hat{G}_{m-1}(z),$$

for $1 \le m \le \ell$. Assume that we have shown $T_{m+1}(z)$ to converge, which holds trivially for $m = \ell$. Then, the right-hand side of this identity is of Gevrey order at most $1/k_{m-1}$, while the left-hand side is k_m-summable, so both sides converge according to Theorem 37 (p. 106). This, however, implies convergence of $T_m(z)$. □

The above theorem shows that FFS with more than one level *cannot be k-summable for any $k > 0$*, because of Theorem 37 (p. 106). Moreover, if only one level occurs, the notions of FFS and HLFFS coincide. In this situation, the summability of FFS was essentially known, but formulated differently, before Ramis created the theory of k-summability.

Exercises: In the following exercises, consider a fixed formal system (3.3) (p. 40) that is elementary.

1. Setting $T_0 = I$, define T_n recursively by

$$-n\, T_n = \sum_{m=0}^{n-1} A_{n+r-m}\, T_m, \quad n \ge 1.$$

   Show that all T_n commute with one another as well as with all A_m, for $m \ge 0$. For $\hat{T}(z) = \sum_0^\infty T_n\, z^{-n}$, conclude formally (3.5) (p. 40) with $\hat{B}(z) = \sum_0^r A_n\, z^{-n}$.

2. Let $A = \Lambda + N$, with a diagonalizable matrix Λ and nilpotent N commuting with Λ (observe that such a decomposition always exists). If B commutes with A, show that B commutes with both Λ and N.

3. Let N be a nilpotent matrix. Show that $T(z) = \exp[m^{-1}\, z^m\, N]$, $m \in \mathbb{N}$, is a meromorphic transformation. In case N commutes with all A_n, show that

$$\hat{B}(z) = T^{-1}(z) \left[\hat{A}(z)\, T(z) - z\, T'(z)\right] = \hat{A}(z) - z^m\, N.$$

4. Given an elementary formal system (3.3), show that there exists a formal meromorphic transformation $\hat{T}(z)$ so that the transformed system has a coefficient matrix $B(z)$ of the form

$$B(z) = \operatorname{diag}\,[b_1(z)\, I_{s_1} + N_1, \ldots, b_\mu(z)\, I_{s_\mu} + N_\mu],$$

   with distinct polynomials $b_m(z)$ of degree at most r satisfying $0 \le \operatorname{Re} b_m(0) < 1$, $1 \le m \le \mu$, and nilpotent constant matrices N_m, whose blocks are increasing in size.

5. For $B(z)$ as above, compute a fundamental solution of the system $z\tilde{x}' = B(z)\,\tilde{x}$.

6. Let N_1, N_2 be nilpotent matrices, not necessarily of the same size. Let $\hat{T}(z)$ be a formal meromophic series satisfying $z\,\hat{T}(z)' = N_1\,\hat{T}(z) - \hat{T}(z)\,N_2$. Show that then $\hat{T}(z)$ is constant.

8.5 Definition of Highest-Level Normal Solutions

After this digression into the classical subject of FFS, we now continue with our investigation of HLFFS. For this purpose, we consider a system (3.1) together with a fixed HLFFS $\hat{F}(z)$, and we shall use the same notation as on p. 55 for its data pairs $(q_j(z), s_j)$, $1 \le j \le \mu$, and the coefficient matrix $B(z)$ of the transformed system. From the data pairs we then can explicitly compute the singular directions for $\hat{F}(z)$ as the solutions of (8.3), with $k = r - p/q$, and we shall from now on denote them by d_j, labeled so that

$$d_{j-1} < d_j,\ j \in \mathbb{Z}, \qquad d_{-1} < 0 \le d_0. \tag{8.8}$$

Interchanging j and m in (8.3), we conclude that d_j is singular if and only if $d_j \pm \pi/k$ is so, too. The data pairs are closed with respect to continuation, so in particular, the set $\{\lambda_n^{(p)},\ 1 \le n \le \mu\}$ is closed with respect to multiplication by $\exp[2\pi i/q]$. Hence, we also see that $d_j \pm 2\pi/(qr-p)$ is again a singular direction, for every integer j. Together, this shows the following:

- Let μ denote the greatest common divisor of q and 2; in other words, $\mu = 2$ whenever q is even, and $\mu = 1$ for odd q. Then for every singular direction d_j, the directions $d_j \pm \mu\pi/(qr-p)$ are singular as well. In other words, every half-open interval of length $\mu\pi/(qr-p)$ contains the same number of singular directions, say, j_1. Set $j_0 = 2(qr-p)\,j_1/\mu$; hence $j_0 = (qr-p)\,j_1$ when q is even, resp. $j_0 = 2(qr-p)\,j_1$ for odd q. Then j_0 is the number of singular directions in every half-open interval of length $2\,\pi$, or equivalently $d_{j\pm j_1} = d_j \pm 2\,\pi$ for every $j \in \mathbb{Z}$.

Let $S_j = \{\, z\ :\ |z| > \rho,\ d_{j-1} - \pi/(2k) < \arg z < d_j + \pi/(2k)\}$. These are sectors near infinity whose boundary rays are usually referred to as *Stokes' directions*[3] because of their significance in the *Stokes phenomenon* (see the following chapter). We choose to denote these directions as $\tau_j = d_j + \pi/(2k)$, $j \in \mathbb{Z}$.

[3] In the literature one sometimes finds the term *anti-Stokes' directions* for what we call the singular directions d_j.

For each d in the interval (d_{j-1}, d_j), the series $\hat{F}(z)$ is going to be k-summable (with $k = r - p/q$), and according to Lemma 10 (p. 101) the sum of $\hat{F}(z)$ is independent of d, but will in general depend on j, hence we choose to denote this sum by $F_j(z)$.

Theorem 44 *The matrix functions $F_j(z)$ defined above are holomorphic for $|z| > \rho$ on the Riemann surface of the logarithm, and there we have*

$$zF_j'(z) = A(z)\,F_j(z) - F_j(z)\,B(z), \quad j \in \mathbb{Z}. \tag{8.9}$$

Moreover, they satisfy $F_j(z) \cong_{1/k} \hat{F}(z)$ for $z \to \infty$ in the sector S_j.

Proof: It follows from the general theory of k-summability that $F_j(z)$ is holomorphic in some sectorial region G at infinity with opening $d_j - d_{j-1} + \pi/k$, and bisecting direction $(d_j + d_{j-1})/2$. Moreover, $F_j(z) \cong_{1/k} \hat{F}(z)$ there. Since $\hat{F}(z)$ formally satisfies

$$z\hat{F}'(z) = A(z)\,\hat{F}(z) - \hat{F}(z)\,B(z),$$

we conclude from the algebraic properties of k-summability that (8.9) holds in G. Since (8.9) is nothing but a system of linear ODE for the entries of $F_j(z)$, we conclude from the general theory in Chapter 1 that $F_j(z)$ can be holomorphically continued along arbitrary paths within the region $\{|z| > \rho\}$, hence is holomorphic on the corresponding part of the Riemann surface of the logarithm. Every closed subsector of S_j with sufficiently large radius is contained in the sectorial region G, so the asymptotic holds in S_j. □

In what follows, we shall use the term *highest-level normal solutions* for the matrices $F_j(z)$ defined above, and we shall use the abbreviation HLNS. We shall study these HLNS in greater detail in the following chapter.

Exercises: In the following exercises we consider two linear systems $z\,x_j' = A_j(z)\,x_j$ for $|z| > \rho$, of arbitrary Poincaré ranks at infinity and of dimension ν_j, with ν_1 not necessarily equal to ν_2.

1. Verify that $z\,X' = A_1(z)\,X - X\,A_2(z)$, $|z| > \rho$, may be rewritten as a linear system of meromorphic ODE in dimension $\nu_1 \cdot \nu_2$.

2. Let $X(z)$ be a solution of the linear system of ODE in the previous exercise, say, near some point z_0, $|z_0| > \rho$. Show that then $X(z)$ can be holomorphically continued along arbitrary paths within $R(\infty, \rho)$. In case $\nu_1 = \nu_2$, show that $\det X(z_0) = 0$ implies $\det X(z) \equiv 0$.

3. For the HLNS, defined above, show that $\det F_j(z) \neq 0$ for arbitrary values z with $|z| > \rho$.

9

Stokes' Phenomenon

The fact that a solution of an ODE, near an irregular singularity, in different sectors of the complex plane in general shows different asymptotic behavior was observed and studied by *Stokes* [258] and is, therefore, named *Stokes' phenomenon.* An equivalent way of describing this phenomenon in the language of summability is as follows: Consider an HLFFS $\hat{F}(z)$, as defined in Section 3.5, of some system (3.1) (p. 37). We have shown $\hat{F}(z)$ to be k-summable, for a unique $k > 0$, in all but some singular directions, thus defining the HLNS $F_j(z)$. In corresponding sectors, all $F_j(z)$ have *the same asymptotic expansion*, despite of the fact that in general they *do* depend upon j: According to Proposition 13 (p. 105), *the matrices $F_j(z)$ are independent of j if and only if the HLFFS $\hat{F}(z)$ converges.* Hence in other words, Stokes' phenomenon is caused by the divergence of formal solutions!

In this chapter, we shall investigate how two consecutive $F_j(z)$ are interrelated, and in this way analyze *Stokes' phenomenon of highest level.* Classically, this part of the theory was based upon the choice of a FFS; see, e.g., [34, 35] or [143]. This has the serious disadvantage of mixing the phenomena arising on different levels, as shall become clearer later on. So this is why we shall work with HLFFS instead.

In principle, Stokes' phenomenon is described by finitely many constant matrices, usually named *Stokes' multipliers*, or *lateral connection matrices.* We shall explicitly describe their structure, and discuss in which sense they can be computed in terms of the HLFFS used in their definition. In some cases of dimension $\nu = 2$, we shall obtain explicit values for the multipliers in terms of the data of the system. We then present a result, first obtained by *Birkhoff* [55, 57], on the *freedom* of the Stokes multipliers. For another proof of this fact using a somewhat different approach, see the monograph of *Sibuya* [251].

Many authors have been concerned with the theoretical investigation and/or numerical computation of Stokes multipliers, e.g., *Turrittin* [267, 272], *Heading* [115, 116], *Okubo* [205], *Hsieh and Sibuya* [130], *Sibuya* [247–250, 252], *Kohno* [154, 155, 160], *Smilyansky* [256], *Braaksma* [66–69], *Gollwitzer and Sibuya* [109], *McHugh* [189], *Jurkat, Lutz, and Peyerimhoff* [146–149], *Emamzadeh* [97–99], *Hsieh* [129], *Schäfke* [237, 240], *Hinton* [121], *Balser, Jurkat, and Lutz* [33, 37–39, 41], *Gurarij and Matsaev* [111], *Ramis* [228, 227], *Balser* [11–13, 16, 20], *Slavyanov* [255], *Schlosser-Haupt and Wyrwich* [243], *Duval and Mitschi* [93], *Babbitt and Varadarajan* [4], *Tovbis* [264, 265], *Lin and Sibuya* [168], *Immink* [135–137], *Tabara* [259], *Kostov* [162], *Naegele and Thomann* [197], *Sibuya and Tabara* [254], and *Olde Dalhuis and Olver* [211]. Also compare Section 12.5 on the so-called *central connection problem.*

9.1 Highest-Level Stokes' Multipliers

Throughout what follows, we consider a fixed system (3.1) (p. 37), having an essentially irregular singularity at infinity. Let $\hat{F}(z)$ be an HLFFS of (3.1), together with the corresponding HLNS $F_j(z)$. Recall that each $F_j(z)$ satisfies (8.9) (p. 138), with $B(z)$ diagonally blocked of type $(s_1, \ldots, s_\mu)$. We arbitrarily choose a fundamental solution $Y(z) = \mathrm{diag}\,[Y_1(z), \ldots, Y_\mu(z)]$ of $z\,y' = B(z)\,y$. It follows from (8.9) that $X_j(z) = F_j(z)\,Y(z)$ is a solution of (3.1). Since $\det X_j(z) \neq 0$ follows from Exercise 3 on p. 138, every $X_j(z)$ is in fact a fundamental solution of (3.1). From the exercises below, we then conclude that *every solution of* (8.9) *can be represented as* $X_j(z)\,C\,Y^{-1}(z)$, *for a suitable constant matrix* C. So in particular, there exist unique matrices V_j such that

$$F_{j-1}(z) - F_j(z) = F_j(z)\,Y(z)\,V_j\,Y^{-1}(z), \quad |z| > \rho, \quad j \in \mathbb{Z}. \tag{9.1}$$

This formula is equivalent to $X_{j-1}(z) = X_j(z)\,(I + V_j)$, for $j \in \mathbb{Z}$, and we shall refer to the matrices V_j as the *Stokes multipliers of highest level* of the system (3.1). Note that V_j depends upon both $\hat{F}(z)$ and $Y(z)$. Therefore, we here shall always consider *pairs* $(\hat{F}(z), Y(z))$, and for simplicity of notation we also use the term HLFFS to refer to such a pair. From (8.9) we conclude that the left-hand side of (9.1) is asymptotically zero of Gevrey order $1/k$ in $S_j \cap S_{j-1}$, and we shall use this to investigate the structure of the Stokes multipliers V_j:

With $q_n(z)$ as in the definition of the data pairs of HLFFS, we set $G_n(z) = \mathrm{e}^{-q_n(z)} Y_n(z)$. From the special form of $B(z)$, and using Exercise 4 on p. 52 together with Exercise 6 on p. 7, *we conclude that each* $G_n(z)$ *and its inverse are of exponential growth strictly less than* k. Hence, the behavior of $Y_n(z)$ near infinity is in principle determined by that of $q_n(z)$. Blocking $V_j = [V_{nm}^{(j)}]$ in the block structure of $Y(z)$, the matrix $Y(z)\, V_j\, Y^{-1}(z)$ on the right-hand side of (9.1) is likewise blocked, and the behavior of a block, as $z \to \infty$, depends upon whether or not we have

$$\exp[q_n(z) - q_m(z)] = \exp[(\lambda_n^{(p)} - \lambda_m^{(p)}) z^k/k] \cong_{1/k} \hat{0} \tag{9.2}$$

for $z \in S_{j-1} \cup S_j$. For a pair (n,m), this holds if and only if $-kd_{j-1} = \arg(\lambda_m^{(p)} - \lambda_n^{(p)}) \bmod 2\pi$. In other words, this characterizes pairs (n,m) for which the left-hand side of (9.2) has maximal rate of descend along the ray $\arg z = d_{j-1}$. The set of all such pairs will be denoted by Supp_j. Note that we always have $(n,n) \notin \mathrm{Supp}_j$, and Supp_j is never empty, owing to the definition of d_{j-1}. The next theorem says that the set of these pairs is "the support" of V_j:

Theorem 45 *For a system* (3.1) *with an essentially irregular singularity at infinity, let* $(\hat{F}(z), Y(z))$ *be a given HLFFS. Then for the corresponding Stokes multipliers of highest level,* $(n,m) \notin \mathrm{Supp}_j$ *implies* $V_{nm}^{(j)} = 0$. *Hence, in particular all the diagonal blocks of* V_j *vanish.*

Proof: From (9.1) we conclude $Y(z)\, V_j\, Y^{-1}(z) \cong_{1/k} \hat{0}$ in $S_{j-1} \cup S_j$. Using Exercise 7 on p. 74, this is equivalent to $\mathrm{e}^{q_n(z)}\, V_{nm}^{(j)}\, \mathrm{e}^{-q_m(z)} = \exp[(\lambda_n^{(p)} - \lambda_m^{(p)})\, z^k]\, V_{nm}^{(j)} \cong_{1/k} \hat{0}$ in $S_j \cap S_{j-1}$, for every (n,m), since $G_n(z)$ and $G_m^{-1}(z)$ both are of exponential growth strictly less than k. According to the definition of Supp_j, this completes the proof. □

Exercises: In the first two exercises to follow, we consider two linear systems $zx_j' = A_j(z)\, x_j$ for $|z| > \rho$, of arbitrary Poincaré ranks at infinity and of dimension ν_j, with ν_1 not necessarily equal to ν_2.

1. Let $X_j(z)$ be a fundamental solution of $zx_j' = A_j(z)\, x_j$. For $C \in \mathbb{C}^{\nu_1 \times \nu_2}$, show that $X_1(z)\, C\, X_2^{-1}(z)$ is a solution of

$$zX' = A_1(z)\, X - X\, A_2(z), \quad |z| > \rho. \tag{9.3}$$

2. Let $X(z)$ be any solution of (9.3). Verify that $X(z)$ can be represented as $X_1(z)\, C\, X_2^{-1}(z)$ for a unique $C \in \mathbb{C}^{\nu_1 \times \nu_2}$.

3. For every pair (n,m), $1 \le n < m \le \mu$, show: Every half-open interval of length π/k contains exactly one direction τ where, for every sufficiently small $\varepsilon > 0$, (9.2) holds for $\arg z = \tau + \varepsilon$ and fails for $\arg z = \tau - \varepsilon$ or vice versa. The set of all these directions, for arbitrary (n,m), are exactly the Stokes directions, i.e., the boundary rays of the sectors S_j.

4. Show that every half-open interval of length π/k contains the same number of Stokes directions, say, n_1. For $\mathrm{Supp}_{j,n_1} = \mathrm{Supp}_{j+1} \cup \ldots \cup \mathrm{Supp}_{j+n_1}$ and distinct $n, m, \ell \in \{1, \ldots, \mu\}$, show:

   (a) $(n,n) \notin \mathrm{Supp}_{j,n_1}$.

   (b) $(n,m) \in \mathrm{Supp}_{j,n_1} \iff (m,n) \notin \mathrm{Supp}_{j,n_1}$.

   (c) $(n,m) \in \mathrm{Supp}_{j,n_1}$ and $(m,\ell) \in \mathrm{Supp}_{j,n_1} \Rightarrow (n,\ell) \in \mathrm{Supp}_{j,n_1}$.

   (d) The matrices of the form $I+C$, with $C_{nm} = 0$ whenever $(n,m) \notin \mathrm{Supp}_{j,n_1}$, form a group $\mathbb{G}_{j,n_1}$ with respect to multiplication.

   (e) Statements (a)–(d), except for the backward direction in (b), also hold for Supp_j instead of Supp_{j,n_1}. In the sequel, denote the corresponding group by $\mathbb{G}_j$.

   (f) For $j+1 \le \ell \le j+n_1$, each $\mathbb{G}_\ell$ is a subgroup of $\mathbb{G}_{j,n_1}$.

   (g) For $I + C \in \mathbb{G}_{j,n_1}$, unique matrices $I + V_\ell \in \mathbb{G}_\ell$ exist with $I + C = (I + V_{j+n_1}) \cdot \ldots \cdot (I + V_{j+1})$.

9.2 The Periodicity Relation

The result we are going to prove next says that, of the infinitely many Stokes multipliers, it suffices to compute only j_0 consecutive ones, for j_0 as defined in Section 8.5. This will be a consequence of the following lemma showing how the multipliers change when we choose another HLFFS – in particular, another fundamental solution $Y(z)$ of the transformed system. Since we have seen that a permutation of the data pairs of an HLFFS $\hat{F}(z)$ reflects in a permutation of its columns, we may without loss of generality restrict to the case of two HLFFS with the same data pairs.

Lemma 12 *Given a system* (3.1) *with an essentially irregular singularity at infinity, let* $(\hat{F}(z;1), Y(z;1))$, $(\hat{F}(z;2), Y(z;2))$ *be two HLFFS with identical data pairs. Then there is a unique constant invertible matrix* $D = \mathrm{diag}\,[D_1, \ldots, D_\mu]$, *diagonally blocked in the block structure determined by the data pairs, for which* $Y(z;1) = T(z)\,Y(z;2)\,D$, *with a q-meromorphic transformation* $T(z)$ *that is likewise blocked and satisfies* $\hat{F}(z;1)\,T(z) = \hat{F}(z;2)$. *For the corresponding Stokes multipliers* $V_j^{(1)}$, $V_j^{(2)}$ *we then have* $V_j^{(1)} = D^{-1}\,V_j^{(2)}\,D$, *for every* $j \in \mathbb{Z}$.

Proof: From the Main Theorem (p. 129) we conclude that the matrix $T(z) = \hat{F}^{-1}(z;1)\,\hat{F}(z;2)$ is diagonally blocked and convergent. If $F_j(z;n)$ denote the HLNS corresponding to $\hat{F}(z;n)$, then $F_j(z;1)\,T(z) = F_j(z;2)$ follows from the properties of k-summability, for every $j \in \mathbb{Z}$. Since the

$F_j(z;n)\,Y(z;n)$ are fundamental solutions of (3.1), for every j and n, we conclude $F_j(z;1)\,Y(z;1) = F_j(z;2)\,Y(z;2)\,C_j$, for constant invertible C_j. This implies $Y(z;1) = T(z)\,Y(z;2)\,C_j$; hence C_j is diagonally blocked and independent of j. Writing D instead of C_j, one may complete the proof. □

Now consider an HLFFS $(\hat{F}(z), Y(z))$. Since its data pairs $(q_j(z), s_j)$ are closed with respect to continuation, there exists a *permutation matrix* R,[1] for which $Q(z) = \mathrm{diag}\,[q_1(z)\,I_{s_1}, \ldots, q_\mu(z)\,I_{s_\mu}] = R\,Q(z\mathrm{e}^{2\pi i})\,R^{-1}$. In fact, R can be made unique by requiring that, when blocked of type $(s_1, \ldots, s_\mu)$, *each block of R is either the zero or the identity matrix of appropriate size.* With this *block permutation matrix R* we now show:

Lemma 13 *Let a system* (3.1) *have an essentially irregular singularity at infinity, and let $(\hat{F}(z), Y(z))$ be an HLFFS. Then there is a unique constant diagonally blocked matrix D and an equally blocked q-meromorphic transformation $T(z)$ for which $Y(z\mathrm{e}^{2\pi i}) = T(z)\,R^{-1}\,Y(z)\,R\,D$, $\hat{F}(z\mathrm{e}^{2\pi i}) = \hat{F}(z)\,R\,T^{-1}(z)$, and $F_{j+j_0}(z\mathrm{e}^{2\pi i}) = F_j(z)\,R\,T^{-1}(z)$, for $j \in \mathbb{Z}$.*

Proof: Observe that $(\hat{F}(z\exp[2\pi i]), Y(z\exp[2\pi i]))$ is again an HLFFS that has the same data pairs as $(\hat{F}(z)\,R, R^{-1}Y(z)\,R)$, and apply Lemma 12 plus the properties of k-summability. □

We denote $R\,D$ by $\exp[2\pi i L]$, and refer to it, resp. to L, as the *formal monodromy factor*, resp. *matrix* of the HLFFS $(\hat{F}(z), Y(z))$.

Proposition 21 *Let a system* (3.1) *have an essentially irregular singularity at infinity, and let $(\hat{F}(z), Y(z))$ be an HLFFS with corresponding Stokes' multipliers V_j, HLNS $F_j(z)$ and fundamental solutions $X_j(z) = F_j(z)\,Y(z)$ of* (3.1). *Then we have for every $j \in \mathbb{Z}$:*

$$X_{j+j_0}(z\mathrm{e}^{2\pi i}) = X_j(z)\,\mathrm{e}^{2\pi i L}, \quad V_{j+j_0} = \mathrm{e}^{-2\pi i L} V_j\,\mathrm{e}^{2\pi i L}. \tag{9.4}$$

Moreover, a monodromy matrix M_j for $X_j(z)$ is given by the identity

$$\mathrm{e}^{2\pi i M_j} = \mathrm{e}^{2\pi i L}(I + V_{j+j_0}) \cdot \ldots \cdot (I + V_{j+1}). \tag{9.5}$$

Proof: From Lemma 13 we obtain (9.4). This in turn implies (9.5), repeatedly using $X_{\ell-1}(z) = X_\ell(z)\,(I + V_\ell)$. □

Note that (9.5) shows that the computation of a monodromy matrix of (3.1) is essentially achieved once we have computed the Stokes multipliers. In some cases both problems even are equivalent, as follows from the exercises below.

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[1] This is a matrix differing from I by a permutation of its columns, or equivalently, rows.

Exercises: In the following exercises, make the same assumptions as in the above proposition. In particular, let matrices always be blocked of type $(s_1, \ldots, s_\mu)$.

1. Prove that if a matrix C *can* be factored as $C = D\,C_+\,C_-$, with D diagonally blocked, and C_+ upper, resp. C_- lower triangularly blocked, both factors having identity matrices along the diagonal, then the factors are uniquely determined.

2. Enumerate the pairs (n, m), $1 \le m < n \le \mu$ in any order, and let all blocks of C on and above the diagonal vanish. Prove $I + C = (I + C_1) \cdot \ldots \cdot (I + C_{\mu(\mu-1)/2})$, with unique matrices C_j whose blocks all vanish except for that one in position (n_j, m_j).

3. Show the following for $k = 1$: If $\exp[2\pi i L]$ and *any one of the matrices* $\exp[2\pi i M_j]$ are known, then *all* the Stokes multipliers V_j can be computed from the identities established above.

9.3 The Associated Functions

Only in a few cases in dimension $\nu = 2$ shall we succeed in expressing the Stokes multipliers in terms of known higher transcendental functions of the parameters of the system; in general their nature seems to be of the same degree of transcendence as that of the solutions itself. Nonetheless, it is of importance to learn as much as possible about how to, at least theroretically, compute them in terms of the data in the system. Here, we shall study in which sense the Stokes multipliers can be obtained via an analysis of the singularities of certain *associated functions*, related to the HLNS by Borel transform. This is a case of what Ecalle has named *resurgent analysis*. While *Braaksma* [69] and *Immink* [137] have obtained equivalent results working with the classical type of FFS, we shall here follow a presentation in [39], which was based on HLFFS, restricted to $k = 1$. For similar results in case of $k = r > 1$, but a leading term with distinct eigenvalues, see *Reuter* [231, 232].

Consider a fixed system (3.1) having an essentially irregular singularity at infinity, and a likewise fixed HLFFS $(\hat{F}(z), Y(z))$. To simplify notation, we shall without loss of generality make the following additional assumptions:

- *Assume that k is an integer.* This can be brought about by a change of variable $z = w^q$ and has the effect that $\hat{F}(z)$ does not contain any roots. By such a change of variable, the Stokes directions τ_j are replaced by τ_j/q, but the Stokes multipliers V_j remain the same. For integer k, the block permutation matrix R introduced above equals I; hence the formal monodromy matrix L is diagonally blocked, say, $L = \operatorname{diag}[L_1, \ldots, L_\mu]$.

- *Assume that the polynomials $q_m(z)$ in the data pairs of the HLFFS have the form $q_m(z) = -u_m\, z^k$, $-u_m = \lambda_m^{(p)}/k$.* This can always be made to hold by a scalar exponential shift. Under this assumption, the transformed system in the definition of HLFFS on p. 56 has Poincaré rank k. Moreover, the diagonal blocks of $Y(z)$ are of the form $Y_m(z) = G_m(z)\exp[-u_m\, z^k]$, with $G_m^{\pm}(z)$ of exponential growth strictly less than k in arbitrary sectors.

- *Assume that $\hat{F}(z)$ is a formal power series with constant term equal to I.* According to Exercise 4 on p. 41, this can be made to hold by means of a terminating meromorphic transformation $T(z)$, which does not change the Stokes multipliers, but may lower the Poincaré rank of the system. Under this assumption we then have that $k = r$, the Poincaré rank of the system (3.1), and this implies $p = 0$.

A system (3.1), satisfying all the above assumptions, will from now on be called *normalized*, and we use the same adjective for an HLFFS of the above form. For such a normalized HLFFS, we shall define *associated functions*. These functions will have branch points at all the points u_m, $1 \le m \le \mu$; hence the corresponding Riemann surface will be somewhat difficult to visualize. Therefore, we shall instead consider a complex u-plane together with cuts from each u_n to infinity along the rays $\arg(u - u_n) = -r\, d$ with arbitrarily fixed d. For simplicity, we choose d so that the cuts are all disjoint. According to the definition of d_j, and observing $u_n = -\lambda_n^{(p)}$, this is so if and only if d is a nonsingular direction for $\hat{F}(z)$ in the sense of Section 6.4. The set of u not on any one of the cuts shall be denoted by $\mathbb{C}_d$. It is worthwhile to mention that, unlike on a Riemann surface, points u here are the same once their arguments differ by a multiple of 2π.

To define the associated functions, we first introduce for every m, with $1 \le m \le \mu$:

$$H_m(u; s; k) = \frac{r}{2\pi i} \int_{\beta(\tau)} z^{s-1} G_m(z)\, e^{z^r (u - u_m)}\, dz, \tag{9.6}$$

for $\tau = -d + (2k+1)\,\pi/r$ with $k \in \mathbb{Z}$, and a path of integration $\beta(\tau)$ consisting of the ray $\arg z = -\tau - (\varepsilon + \pi)/(2r)$ from infinity to the point z_0 with $|z_0| = \rho + \varepsilon$, then on the circle $|z| = \rho + \varepsilon$ to the ray $\arg z = -\tau + (\varepsilon + \pi)/(2r)$, and back to infinity along this ray, for small $\varepsilon > 0$. Note that the integral is independent of ε, and absolutely and compactly convergent for $s \in \mathbb{C}$ and $|\arg(u - u_m) - r\,\tau| < \varepsilon/2$. After changes of variable $z \mapsto 1/z$ and $u \mapsto u^r$, the integral (9.6) is nothing but the Borel transform[2] of $z^s\, Y_m(z)$.

[2] Observe that here it is important to take the original form of the Borel operator, as defined on p. 80.

Because of the choice of τ, the sector of convergence of (9.6) is a subset of $\mathbb{C}_d$. Turning the path of integration by an angle of α, i.e., replacing τ by $\tau - \alpha$, obviously results in continuation of $H_m(u; s; k)$ in the opposite sense by the angle $-r\alpha$. In particular, the points u_n for $n \neq m$ do not cause any problems as regards continuation of the function! Therefore, each branch $H_m(u; s; k)$ becomes a holomorphic function in $\mathbb{C}_d$, *and is even holomorphic at every point u on the cuts* $\arg(u - u_n) = -r\,d$, *including the point u_n itself, for every $n \neq m$.* As a convenient way of stating this, we shall write $H_m(u; s; k) = \text{hol}(u - u_n)$ for $n \neq m$, $1 \leq n, m \leq \mu$. Moreover, the various branches $H_m(u; s; k)$ are interrelated as follows: The values of $H_m(u; s; k)$ on the right-hand side of the mth cut[3] equal those of $H_m(u; s; k+1)$ on the left-hand side. So in other words, *holomorphic continuation of $H_m(u; s; k)$ across the cut in the positive direction produces the branch $H_m(u; s; k+1)$.* Moreover, from $Y(z\,e^{2\pi i}) = Y(z)\exp[2\pi i\,L_m]$ we conclude via a corresponding change of variable in (9.6) that $H_m(u; s; k + r) = H_m(u; s; k)\,e^{-2\pi i\,(sI + L_m)}$. This shows that r *consecutive branches generate the others via explicit linear relations.*

Block the formal series $\hat{F}(z) = \sum_0^\infty F_n\,z^{-n}$ into *column blocks* $\hat{F}_m(z) = \sum_0^\infty F_{n,m}\,z^{-n}$ of s_m consecutive columns of $\hat{F}(z)$. The series

$$\Phi_m(u; s; k) = \sum_{n=0}^{\infty} F_{n,m}\,H_m(u; s - n; k), \tag{9.7}$$

according to Exercise 2 and the Gevrey order of $\hat{F}(z)$, converges absolutely for $s \in \mathbb{C}$ and $0 < |u - u_m| < \delta$, with sufficiently small $\delta > 0$, and convergence is locally uniform in u. Following the terminology introduced in [39], we shall name $\Phi_m(u; s; k)$ *the associated functions to the HLFFS* $(\hat{F}(z), Y(z))$. In a way, the series (9.7) is the formal Borel transform of index r of $\hat{F}_m(z)\,Y_m(z)$. As we shall show in Lemma 14, this analogy is more than formal, since $\Phi_m(u; s; k)$ will turn out to be the Borel transform of the corresponding column blocks of the HLNS $X_j(z)$.

Note that (9.7) *in principle allows to compute the associated functions* $\Phi_m(u; s; k)$ *in terms of the HLFFS* $(\hat{F}(z), Y(z))$ *without knowledge of the HLNS.* Thus, we take the attitude of *the matrices* $\Phi_m(u; s; k)$ *as being known*, at least near the point u_m. From the above monodromy behavior of $H_m(u; s; k)$ we immediately obtain that $\Phi_m(u; s; k+1)$ is the holomorphic continuation of $\Phi_m(u; s; k)$ across the mth cut, in the positive sense, while

$$\Phi_m(u; s; k + r) = \Phi_m(u; s; k)\,e^{-2\pi i\,(sI + L_m)}. \tag{9.8}$$

To discuss holomorphic continuation of these functions into $\mathbb{C}_d$, we first establish an integral representation:

[3] To understand this or any statement like "a point is to the right of the mth cut," rotate $\mathbb{C}_d$ about the origin so that the cuts point "upwards," in which case the meaning of "left" and "right" is intuitive.

For $\tau = -d + (2k+1)\,\pi/r$ with $k \in \mathbb{Z}$, there is a unique $j = j(k) \in \mathbb{Z}$ for which $d_{j-1} < -\tau < d_j$. By definition of the number n_1 in Exercise 4 on p. 142, we have $j(k+1) = j(k) - 2\,n_1$ for every k. With this value $j(k)$, consider the integrals

$$\frac{r}{2\pi i} \int_{\beta(\tau)} z^{s-1} F_{j(k),m}(z)\, G_m(z)\, e^{z^r (u-u_m)}\, dz, \quad 1 \le m \le \mu, \tag{9.9}$$

with $\beta(\tau)$ as in (9.6). Note that for sufficiently small $\varepsilon > 0$, the path $\beta(\tau)$ fits into the sector $S_{j(k)}$ where $F_{j(k),m}(z)$ is bounded. So the integral converges for $s \in \mathbb{C}$ and $|\arg(u - u_m) - r\,\tau| < \varepsilon/2$ and is holomorphic in both variables. By variation of τ, we can continue the function with respect to u into the region $G_{k,m} = \{z : \; -r\,d_{j(k)} < \arg(u - u_m) < -r\,d_{j(k)-1}\} \subset \mathbb{C}_d$. As we shall show now, this integral represents $\Phi_m(u; s; k)$:

Lemma 14 *Under the assumptions made above, the integral* (9.9) *equals* $\Phi_m(u; s; k)$, *for arbitrary* $s \in \mathbb{C}$ *and* $u \in G_{k,m}$ *with* $|u - u_m|$ *sufficiently small. Hence* $\Phi_m(u; s)$ *is holomorphic in* $G_{k,m}$.

Proof: Fix a value for τ and restrict to such u with $\arg(u - u_m) = r\,\tau$. For $N \in \mathbb{N}$, let $\Phi_m(u; s; k; N)$ stand for the difference between the integral and the first N terms of the right-hand side of (9.7). Observing (9.6), express $\Phi_m(u; s; k; N)$ as a corresponding integral. By estimates completely analogous to those in the proof of Theorem 23 (p. 80), one can show that $\Phi_m(u; s; k; N)$ tends to zero as $N \to \infty$, for sufficiently small values of $|u - u_m|$. The identity theorem then completes the proof. □

To discuss the global behavior of $\Phi_m(u; s; k)$ in $\mathbb{C}_d$, we introduce *auxiliary functions* $\Phi^*_m(u; s; k)$ as follows: For every $k \in \mathbb{Z}$, there exists a unique $j = j^*(k) \in \mathbb{Z}$ with $d_{j-1} < d - 2k\,\pi/r < d_j$, and we have $j^*(k) = j(k) + n_1$, with $j(k)$ and n_1 as above. With this value $j^*(k)$ and any fixed point z_0 of modulus larger than ρ and independent of k, we define

$$\Phi^*_m(u; s; k) = \frac{r}{2\pi i} \int_{z_0}^{\infty(\alpha)} z^{s-1} F_{j^*(k),m}(z)\, G_m(z)\, e^{z^r (u-u_m)}\, dz,$$

integrating along the circle $|z| = |z_0|$ to the ray $\arg z = \alpha$, and then along this ray to infinity. For $d_{j^*(k)-1} - \pi/(2r) < \alpha < d_{j^*(k)} + \pi/(2r)$, this integral converges compactly for $s \in \mathbb{C}$ and $\pi/2 < r\,\alpha + \arg(u - u_m) < 3\pi/2$. Varying α, the function can be continued into $-r\,d_{j^*(k)} < \arg(u - u_m) < 2\pi - r\,d_{j^*(k)-1}$. By definition of $j^*(k)$, we therefore see that $\Phi^*_m(u; s; k)$ *is holomorphic in* $\mathbb{C}_d$, *and even holomorphic at points on the nth cut, for every* $n \ne m$, including the point u_n itself. Using the same notation as above, we write $\Phi^*_m(u; s; k) = \mathrm{hol}(u - u_n)$, for $n \ne m$, $1 \le n, m \le \mu$. Unlike the associated functions, the auxiliary ones are not directly linked via holomorphic continuation across the mth cut. However, from (9.4) one obtains that

$$\Phi_m^*(u;s;k+r) = \Phi_m^*(u;s;k)\,\mathrm{e}^{-2\pi i(sI+L_m)}, \quad u \in \mathbb{C}_d. \tag{9.10}$$

The $\Phi_m^*(u;s;k)$ are linked to the auxiliary functions as follows:

Lemma 15 *For every $k \in \mathbb{Z}$ and every m with $1 \le m \le \mu$, the associated function $\Phi_m(u;s;k)$ is holomorphic in $\mathbb{C}_d$. Moreover, we have*

$$\Phi_m(u;s;k) = \Phi_m^*(u;s;k) - \Phi_m^*(u;s;k+1) + \mathrm{hol}(u-u_m).$$

Proof: Let τ and $j(k)$ be as in (9.9). The sectors $G_{k,m}$ all have parallel bisecting rays, so their intersection G_k is not empty. For $u \in G_k$, let $\Phi(u;s;k) = [\Phi_1(u;s;k),\ldots,\Phi_\mu(u;s;k)]$, and define $\Phi^*(u;s;k)$ analogously. From (9.9) we then conclude

$$\Phi(u;s;k) = \frac{r}{2\pi i}\left\{\int_{z_0}^{\infty(\alpha_+)} - \int_{z_0}^{\infty(\alpha_-)}\right\} z^{s-1}\,X_{j(k)}(z)\,\mathrm{e}^{z^r u}\,dz,$$

with $\alpha_\pm = -\tau \pm (\pi+\varepsilon)/(2r)$, and $X_{j(k)}(z)$ as in Section 9.1. While the two paths of integration have the same form as in the definition of the auxiliary functions, the index $j(k)$ is not correct: There we may only use values j, differing from $j(k)$ by odd multiples of n_1, and so that the path of integration fits into S_j. For α^+, this is correct for $j^*(k) = j(k)+n_1$, while for α^- this is so with $j^*(k+1) = j(k)-n_1$. According to the definition of the Stokes multipliers, $X_{j(k)}(z) = X_{j^*(k)}(z)\,(I+C_k^+) = X_{j^*(k+1)}(z)\,(I+C_k^-)$, with $I+C_k^+ = (I+V_{j(k)+n_1})\cdot\ldots\cdot(I+V_{j(k)+1})$, $(I+C_k^-)^{-1} = (I+V_{j(k)})\cdot\ldots\cdot(I+V_{j(k)-n_1+1})$. Consequently we have

$$\Phi(u;s;k) = \Phi^*(u;s;k)\,(I+C_k^+) - \Phi^*(u;s;k+1)\,(I+C_k^-). \tag{9.11}$$

This implies holomorphy of $\Phi(u;s;k)$ in $\mathbb{C}_d$. From Exercise 4 on p. 142 we obtain that both matrices C_k^+ and C_k^- have vanishing diagonal blocks. Since $\Phi_n^*(u;s;k)$ and $\Phi_n^*(u;s;k+1)$ both are holomorphic at u_m whenever $n \ne m$, the proof is completed. □

Formula (9.11) contains more than the information we have used to prove the lemma: Let us write $m \prec n$, *whenever the nth cut is located to the right of the mth one.*[4] Consider any $(n,m) \in \mathrm{Supp}_{j(k),n_1}$, i.e., in view of Exercise 4 on p. 142, any $(n,m) \in \mathrm{Supp}_\ell$ with $j(k)+1 \le \ell \le j(k)+n_1$. From the definition on p. 141, we conclude that this holds if and only if $\arg(u_n - u_m) = -r\,d_{\ell-1}$. By definition of $j(k)$, one finds that the point u_n, when pictured in $\mathbb{C}_d$, is located in the sector $-r\,d-\pi < \arg(u-u_m) < -r\,d$.

[4] As explained earlier, this is to be understood after a rotation of $\mathbb{C}_d$, which makes the cuts point upward.

So $(n,m) \in \mathrm{Supp}_{j(k),n_1}$ *holds if and only if* $n \prec m$. Similarly, one finds $(n,m) \in \mathrm{Supp}_{j(k)-n_1,n_1}$ if and only if $m \prec n$. In both cases, we write $C^{(k)}_{n,m}$ for the corresponding block of C^+_k, resp. C^-_k. Using this terminology, we state the following central result for the behavior of the associated functions at the points u_n:

Theorem 46 *Let a normalized system* (3.1) *with a normalized HLFFS* $(\hat{F}(z), Y(z))$ *be given, and let d be a nonsingular direction. For every* $n \neq m$ *and every* $s \in \mathbb{C}$ *for which* $(I - \mathrm{e}^{2\pi i\,(sI+L_n)})^{-1}$ *exists, the corresponding associated functions satisfy*

$$\Phi_m(u;s;k) = \left[\sum_{\ell=0}^{r-1} \Phi_n(u;s;k+\ell)\right] (I - \mathrm{e}^{-2\pi i\,(sI+L_n)})^{-1}\, C^{(k)}_{nm} + \mathrm{hol}(u-u_n),$$

whenever $n \prec m$ *in the above sense, while in the opposite case we have*

$$\Phi_m(u;s;k) = \left[\sum_{\ell=1}^{r} \Phi_n(u;s;k+\ell)\right] (\mathrm{e}^{-2\pi i\,(sI+L_n)} - I)^{-1}\, C^{(k)}_{nm} + \mathrm{hol}(u-u_n).$$

Proof: For the proof, we shall restrict ourselves to the first case; the arguments in the second are very much analogous. Recall that $\Phi^*_n(u;s;k)$ is holomorphic everywhere except at u_n. Using this, together with the form of $C^{\pm}_k$, we conclude from (9.11) that $\Phi_m(u;s;k) = \Phi^*_n(u;s;k)\, C^{(k)}_{nm} + \mathrm{hol}(u-u_n)$. From Lemma 15 and (9.10) we conclude $\sum_{\ell=0}^{r-1} \Phi_n(u;s;k+\ell) = \Phi^*_n(u;s;k) - \Phi^*_n(u;s;k+r) + \mathrm{hol}(u-u_n) = \Phi^*_n(u;s;k)\,(I - \mathrm{e}^{-2\pi i\,(sI+L_n)}) + \mathrm{hol}(u-u_n)$, completing the proof. □

This theorem has the following consequence:

Corollary to Theorem 46 *Under the assumptions of the above theorem, let* $\Phi_{m,n}(u;s;k)$ *stand for the continuation of* $\Phi_m(u;s;k)$ *across the nth cut in the positive, resp. negative, sense whenever* $n \prec m$, *resp.* $m \prec n$. *Then in both cases*

$$\Phi_m(u;s;k) - \Phi_{m,n}(u;s;k) = \Phi_n(u;s;k)\, C^{(k)}_{nm}. \tag{9.12}$$

Proof: Observe that $\Phi_n(u;s;k+\ell)$, when continued across the nth cut in the positive sense, becomes $\Phi_n(u;s;k+1+\ell)$, while terms holomorphic at u_n remain unchanged. □

The above identities theoretically may be used to compute all Stokes' multipliers of highest level. This shall be investigated in some detail in Section 9.5.

Exercises: In what follows, consider a fixed normalized system (3.1) and a likewise fixed normalized HLFFS $(\hat{F}(z), Y(z))$, and observe the notation introduced above. Moreover, let $B_n = \operatorname{diag}[B_n^{(1)}, \ldots, B_n^{(\mu)}]$ be the coefficients of the transformed system in the definition of HLFFS on p. 56.

1. For fixed $u \in \mathbb{C}_d$, $k \in \mathbb{Z}$, and $1 \le m \le \mu$, show that $H_m(u; s; k)$ satisfies the following difference equation in the variable $s \in \mathbb{C}$: $s\, H_m(u; s; k) + ru\, H_m(u; s+r; k) = -\sum_{n=0}^{n_0} B_n^{(m)}\, H_m(u; s+r-n; k)$.

2. For m, s, and k as above, and arbitrary $\delta > 0$, show existence of $c, K > 0$ so that $\|H_m(u; s-n; k)\| \le c\,(K|u-u_m|)^{n/r}/\Gamma(1+n/r)$ for every $n \in \mathbb{N}_0$ and every $u \in \mathbb{C}$ with $|u - u_m| \ge \delta$.

9.4 An Inversion Formula

Since the associated functions have been shown to be the Borel transform of some of the normal solutions, we may ask for a corresponding inversion formula. However, in general this will not be a normal type of Laplace integral, owing to the kind of singularity of $\Phi_m(u; s; k)$ at u_m. So instead of a ray, we shall use a path of integration $\gamma_m(d)$, very much like the one in *Hankel's formula*: From infinity along the left border of the mth cut until some point close to u_m, then on a small, positively oriented circle around u_m, and back to infinity along the right border of the same cut. With this path, consider the integral

$$\int_{\gamma_m(d)} \left[\sum_{\ell=0}^{r-1} \Phi_m(u; s; k+\ell)\right] \mathrm{e}^{-z^r(u-u_m)}\, du. \tag{9.13}$$

From the definition of the auxiliary functions we conclude that they are of exponential growth at most r everywhere in $\mathbb{C}_d$, including the boundaries of the cuts. From (9.11) we obtain that the same is true for the associated functions. This then shows that the integral converges in a sectorial region G at infinity of opening π/r and bisecting direction d. There we can identify the integral with one of the normal solutions:

Theorem 47 *Let a system* (3.1) *with a normalized HLFFS* $(\hat{F}(z), Y(z))$ *be given, and let d be a nonsingular direction. Then for $k \in \mathbb{Z}$ and $1 \le m \le \mu$, the integral* (9.13) *equals* $z^{s-r}\, F_{j^*(k),m}(z)\, G_m(z)\, (I - \mathrm{e}^{-2\pi i(sI+L_m)})$ *on the region G.*

Proof: Recall $\sum_{\ell=0}^{r-1} \Phi_n(u; s; k+\ell) = \Phi_n^*(u; s; k)\,(I - \mathrm{e}^{-2\pi i\,(sI+L_n)}) + \mathrm{hol}(u - u_n)$ from the proof of Theorem 46. Then, use Exercise 2 and observe that an integral $\int_{\gamma_m(d)} \Phi(u)\, \mathrm{e}^{-z^r(u-u_m)}\, du$ vanishes whenever $\Phi(u)$ is holomorphic along the path as well as inside the circle about u_m inside of $\gamma_m(d)$. □

The above inversion formula is interesting in its own right but will mainly serve in the next section to show that the columns of $\Phi_m(u; s, k)$ are linearly independent.

Exercises: In what follows, make the same assumptions as in the above theorem.

1. For $z_0 \in S_{j*(k)}$, rewrite the integral representation of $\Phi^*_m(u; s; k)$ as a Laplace integral of order 1.
2. Show $z^{s-r}\, F_{j^*(k),m}(z)\, G_m(z) = \int_{\gamma_m(d)} \Phi^*_m(u; s; k)\, \mathrm{e}^{-z^r(u-u_m)}\, du$.

9.5 Computation of the Stokes Multipliers

At the end of the previous section we obtained several identities showing how the behavior of the associated functions at the singularities u_n can be described in terms of the Stokes multipliers of (3.1) (p. 37). Here, we wish to show that this behavior in turn *determines* all the Stokes multipliers. To do this, we first prove that (9.12) determines the matrices $C^{(k)}_{nm}$ uniquely, owing to linear independence of the columns of $\Phi_m(u; s; k)$.

Lemma 16 *Let a system* (3.1) *with a normalized HLFFS* $(\hat{F}(z), Y(z))$ *be given, and let d be a nonsingular direction. Then for every $m = 1, \ldots, \mu$, $k \in \mathbb{Z}$, and $s \in \mathbb{C}$ so that $I - \exp[-2\pi i(sI + L_m)]$ is invertible, the following is correct: If $\Phi_m(u; s; k)\, c = \mathrm{hol}(u - u_m)$ for some $c \in \mathbb{C}^{\,s_m}$, then $c = 0$ follows. In particular, the columns of $\Phi_m(u; s; k)$ are linearly independent functions of $u \in \mathbb{C}_d$.*

Proof: Recall that $\Phi_m(u; s; k+\ell)$ are all interrelated by continuation across the mth cut. Therefore, $\Phi_m(u; s; k)\, c = \mathrm{hol}(u - u_m)$ implies the same with $k + \ell$ instead of k. Consequently, owing to Theorem 47, we obtain $F_{j^*(k),m}(z)\, G_m(z)\, (I - \mathrm{e}^{-2\pi i(sI+L_m)})\, c \equiv 0$. Since $F_{j^*(k),m}(z)$ has linearly independent columns, this implies $c = 0$. □

Examples show that invertibility of $I - \exp[-2\pi i(sI + L_m)]$ is not necessary for the linear independence of the columns of $\Phi_m(u; s; k)$. Whether or not they can ever be linearly dependent seems to be unknown, but will not be of importance here.

The above lemma guarantees that the matrices $C^{(k)}_{nm}$, for every $k \in \mathbb{Z}$ and $n \neq m$, $1 \leq n, m \leq \mu$, are determined by (9.12), or even by the identities in Theorem 46. Exercise 4 on p. 142 shows that from the matrices $C^{\pm}_k$ we can compute the Stokes multipliers V_ℓ, for $j(k)-n_1+1 \le \ell \le j(k)+n_1$. Doing so for r consecutive values of k then gives enough multipliers to compute all others with help of (9.4). Hence, in principle the problem of computing the Stokes multipliers of highest level has been solved. In special situations, however, there are more effective formulas for this computation.We shall briefly illustrate this in the case of Poincaré rank $r=1$ and the leading term having all distinct eigenvalues. In this situation, the following holds:

- The notions of HLFFS and FFS coincide, since there is only one level to consider.
- There are ν distinct values u_n, equal to the negative of the eigenvalues of the leading term A_0.
- Owing to $r=1$ and the form of a FFS, as stated in Exercise 4 on p. 45, the associated functions here are vectors given by convergent power series $\Phi_m(u;s;k) = \sum_0^\infty F_{n,m}\,(u-u_m)^{n-\ell_m-s}/\Gamma(1+n-\ell_m-s)$, with not necessarily distinct complex numbers ℓ_m and $-d+2k\pi < \arg(u-u_m) < -d+2(k+1)\pi$; for this, compare Exercise 1.
- The formal monodromy matrices here are scalar, and $L_m = \ell_m$. The vector $F_{0,m}$ equals the mth unit vector e_m.

In this situation, Theorem 46 states that for $n \prec m$

$$\Phi_m(u;s;k) = \Phi_n(u;s;k)\,(I - \mathrm{e}^{-2\pi i\,(sI+\ell_n)})^{-1}\,C_{nm}^{(k)} + \mathrm{hol}(u-u_n).$$

For $\mathrm{Re}\,(\ell_n+s) > 0$, a term $(u-u_n)^{\ell_n+s}\,\Phi(u)$ tends to 0 whenever $\Phi(u)$ is holomorphic at u_n. Hence we may evaluate the scalar blocks $C_{nm}^{(k)}$ by finding the limit of $(u-u_n)^{\ell_n+s}\,\Phi_m(u;s;k)$ when $u \to u_n$. If $\arg(u-u_n)$ is chosen according to $-d+2k\pi < \arg(u-u_n) < -d+2(k+1)\pi$, this limit equals, using (B.14) (p. 232):

$$\frac{e_n\,c_{nm}^{(k)}}{(1-\mathrm{e}^{-2\pi i(s+\ell_n)})\,\Gamma(1-s-\ell_n)} = \frac{c_{nm}^{(k)}\,\Gamma(s+\ell_n)}{2\pi i}\,\mathrm{e}^{\pi i(s+\ell_n)}\,e_n.$$

Setting $(u-u_n)^{\ell_n+s} = (u_n-u)^{\ell_n+s}\,\mathrm{e}^{\pi i(s+\ell_n)}$, we find

$$\lim_{u\to u_n}(u_n-u)^{\ell_n+s}\,\Phi_m(u;s;k) = \frac{c_{nm}^{(k)}\,\Gamma(s+\ell_n)}{2\pi i}\,e_n,$$

with $-d+(2k-1)\pi < \arg(u_n-u) < -d+(2k+1)\pi$. A similar formula may be obtained for $m \prec n$ as well. In Exercise 2 we shall use this result to find explicit values for the Stokes multipliers of the two-dimensional confluent hypergeometric system.

Exercises: Consider a fixed normalized HLFFS of a system (3.1) and a nonsingular direction d.

1. Assume $G_m(z) \equiv I_{s_m}$, for some m, $1 \le m \le \mu$. Show $\Phi_m(u; s; k) = \sum_0^\infty F_{n,m}\,(u - u_m)^{(n-s)/r}/\Gamma(1+(n-s)/r)$, for $u \in \mathbb{C}_d$, with the branch of $(u - u_m)^{-s/r}$ determined according to $-r\,d < \arg(u - u_m) - 2k\pi i < r - d + 2\pi$.

2. For $A(z)$ as in Exercise 2 (a) on p. 58, compute the associated functions in terms of hypergeometric ones, and find all Stokes multipliers.

3. Consider a confluent hypergeometric system (2.5) (p. 21), with A having distinct eigenvalues. Show that the associated functions then satisfy the hypergeometric system $(u\,I + A)\,\phi' = -(s\,I + B)\,\phi$.

9.6 Highest-Level Invariants

In this section, we shall briefly discuss the notion of equivalence of systems of meromorphic ODE. This concept was first introduced and studied by *Birkhoff* [54] in his attempt to classify such systems with respect to the behavior of solutions near infinity: Throughout, we consider two ν-dimensional systems

$$z\,x' = A(z)\,x, \qquad z\,y' = B(z)\,y, \qquad |z| > \rho, \tag{9.14}$$

with holomorphic coefficient matrices, each having a pole at infinity of, possibly distinct, order r_A resp. r_B. These two systems are said to be *analytically*, resp. *meromorphically, equivalent* to one another, if there exists an analytic, resp. meromorphic, transformation $T(z)$ satisfying

$$zT'(z) = A(z)\,T(z) - T(z)\,B(z), \qquad |z| > \rho. \tag{9.15}$$

Indeed, this is an equivalence relation for meromorphic systems near infinity, and solutions of equivalent systems essentially behave alike at infinity – however, note that in case of meromorphic equivalence the behavior agrees only up to integer powers of z.

Birkhoff introduced the above notion of equivalence in connection with the following approach toward analyzing the behavior near infinity of solutions of systems (3.1). Imagine that we have succeeded in completing the following two tasks:

- Find a collection of objects, named *analytic resp. meromorphic invariants*, which can in some sense be computed in terms of an arbitrarily given system (3.1). These invariants should be such that for each two systems that are analytically resp. meromorphically equivalent all these objects agree, which explains their name. Moreover, the collection should be *complete* in the sense that any two systems sharing the same invariants are indeed analytically, resp. meromorphically equivalent. In other words, *the system of analytic resp. meromorphic invariants characterizes the corresponding equivalence class of systems* (3.1).

- Within each equivalence class of systems with respect to analytic resp. meromorphic equivalence, find a unique *representative* that, in some sense or another, is *the simplest system* in this class, and study the behavior of its solutions near infinity.

Assuming that the above has been done, one can then completely analyse the behavior of solutions of an arbitrary system by *first computing its invariants, thus, determining the equivalence class to which the system belongs, and then identifying the corresponding representative for this equivalence class* – then, the solution of the given system behave as the ones for the corresponding representative, and their behavior is known!

While Birkhoff himself found a complete system of invariants only under some restrictive assumptions, the general case has been treated much later by *Balser, Jurkat, and Lutz* [33–36, 41]. Also compare *Sibuya* [250], or *Lutz and Schäfke* [177]. Here we shall present a simple result on what we call *highest-level meromorphic invariants*. To do so, assume that both systems (9.14) have an essentially irregular singular point at infinity. Let $(\hat{F}_A(z), Y_A(z))$ and $(\hat{F}_B(z), Y_B(z))$ be HLFFS of the corresponding system, with corresponding Stokes' multipliers $V_{j,A}$ and $V_{j,B}$ and formal monodromy matrices L_A and L_B.

Theorem 48 *Let two meromorphic systems* (9.14) *be given.*

(a) *Assume the systems to be meromorphically equivalent. Then the data pairs of the two HLFFS* $(\hat{F}_A(z), Y_A(z))$ *and* $(\hat{F}_B(z), Y_B(z))$ *coincide up to a renumeration.*

(b) *Assume that the data pairs of* $(\hat{F}_A(z), Y_A(z))$ *and* $(\hat{F}_B(z), Y_B(z))$ *coincide. Then the systems* (9.14) *are meromorphically equivalent if and only if there exists a constant invertible matrix* D*, diagonally blocked of type* $(s_1, \ldots, s_\mu)$*, so that*

$$V_{j,A} = D^{-1}\, V_{j,B}\, D, \quad j \in \mathbb{Z}, \qquad \mathrm{e}^{2\pi i\, L_A} = D^{-1}\, \mathrm{e}^{2\pi i\, L_B}\, D, \quad (9.16)$$

and in addition $Y_A(z)\, D^{-1}\, Y_B^{-1}(z)$ *has moderate growth, for* $z \to \infty$*, in arbitrary sectors.*

Proof: In case of meromorphic equivalence, there exists a meromorphic transformation $T(z)$ with (9.15). Hence $(T(z)\, \hat{F}_B(z), Y_B(z))$ is an HLFFS of the first system in (9.14), which implies (a). In case of identical data pairs, we conclude from Lemma 12 (p. 142) existence of diagonally blocked matrices D, constant invertible, and $T_q(z)$, a q-meromorphic transformation, for which $Y_A(z) = T_q(z)\, Y_B(z)\, D$, $\hat{F}_A(z)\, T_q(z) = T(z)\, \hat{F}_B(z)$, and the relation for the Stokes multipliers follow. The first identity implies moderate growth of $Y_A(z)\, D^{-1}\, Y_B^{-1}(z)$, while the second one, owing to properties of k-summability, shows $F_{j,A}(z)\, T_q(z) = T(z)\, F_{j,B}(z)$ for the HLNS, for every $j \in \mathbb{Z}$. This shows $X_{j,A}(z) = T(z)\, X_{j,B}(z)\, D$ for every $j \in \mathbb{Z}$, implying $\exp[2\pi i\, M_{j,A}] = D^{-1} \exp[2\pi i\, M_{j,B}]\, D$. This, together with (9.5) and the relation between the Stokes multipliers, then implies $\exp[2\pi i\, L_A] = D^{-1} \exp[2\pi i\, L_B]\, D$.

Conversely, *define* $T(z) = X_{j,A}(z)\, D^{-1}\, X_{j,B}^{-1}(z) = F_{j,A}(z)\, T_q(z)\, F_{j,B}^{-1}(z)$, with $T_q(z) = Y_A(z)\, D^{-1}\, Y_B^{-1}(z)$. The relation for the Stokes multipliers shows that $T(z)$ is independent of j, and from (9.5) and $\exp[2\pi i\, L_A] = D^{-1} \exp[2\pi i\, L_B]\, D$ we obtain $T(z e^{2\pi i}) = T(z)$. Moreover, moderate growth of $T_q(z)$, together with the asymptotic of both $F_{j,A}(z)$ and $F_{j,B}(z)$, shows that $T(z)$ is of moderate growth in every sector S_j. Consequently, $T(z)$ can only have a pole at infinity; hence is a meromorphic transformation. □

While the Stokes multipliers correspond uniquely to a selected HLFFS of (3.1), there is a freedom in selecting this HLFFS, and this freedom exactly reflects in a change of the Stokes multipliers as in (9.16). Thus, we may also say as follows: Under the assumptions of the above theorem, *the systems* (9.14) *are meromorphically equivalent if and only if we can select corresponding HLFFS of each system so that their Stokes' multipliers and formal monodromy factors agree and $Y_A(z)\, Y_B^{-1}(z)$ is of moderate growth.*

Exercises:

1. Show that the notion of analytic, resp. meromorphic, equivalence of systems is indeed an equivalence relation.

2. For analytic equivalence, verify that the spectrum of the leading term A_0 is invariant. In case of meromorphic equivalence, show that the same is correct, once the transformation does not change the Poincaré rank of the system.

9.7 The Freedom of the Highest-Level Invariants

In Section 9.2 we have shown the Stokes multipliers to satisfy the relations (9.4). In this section we shall show that, aside from the restriction of their support, nothing more can be said in general. For this purpose, we shall prove a technical lemma which is a simplified version of a result of

Sibuya's [251, Section 6.5]: Consider a closed sector $\bar S(\rho) = \{\, z \,:\, |z| \ge \rho,\ \alpha \le \arg z \le \beta \,\}$, with $\alpha < \beta$ fixed. Let $X(z)$ be a $\nu \times \nu$-matrix, holomorphic in the interior of $\bar S(\rho)$ and continuous up to its boundary, and so that $\|X(z)\| \le c\,|z|^{-2}$ in $\bar S(\rho)$. Finally, define $S^*(\tilde\rho, \delta) = \{\, z \,:\, |z| > \tilde\rho, \alpha + \delta < \arg z < \beta + 2\pi - \delta \,\}$, with $\delta > 0$, and $\tilde\rho \ge \rho$ to be determined.

Lemma 17 *Under the above assumptions, for every $\delta > 0$ and sufficiently large $\tilde\rho \ge \rho$, there is a matrix $T(z)$, holomorphic in $S^*(\tilde\rho, \delta)$, and tending to 0 as $z \to \infty$ there, so that:*

$$T(ze^{2\pi i}) - T(z) = (I + T(z))\, X(z), \quad z \in \bar S(\rho) \cap S^*(\tilde\rho, \delta).$$

Proof: For the proof, we proceed exactly as in [251, pp. 152–162]: For $m \in \mathbb{N}_0$, set $\bar S_m = \{\, z \,:\, |z| \ge \tilde\rho + \delta_m,\ \alpha + \delta_m \le \arg z \le \beta - \delta_m \,\}$, $\bar S^*_m = \{\, z \,:\, |z| \ge \tilde\rho + \delta_m,\ \alpha + \delta_m \le \arg z \le \beta - \delta_m + 2\pi \,\}$, with $\delta_m = (1 - 2^{-m})\,\delta$. Beginning with $X_0(z) = X(z)$, define inductively

$$\begin{aligned} U_m(z) &= \frac{-1}{2\pi i} \int_{z_m}^{\infty} X_m(u) \frac{du}{u - z}, \\ X_{m+1}(z) &= U_m(z)\, X_m(z)\, [I + U_m(ze^{2\pi i})]^{-1}, \end{aligned}$$

where $z_m = (\tilde\rho + \delta_m) \exp[i(\alpha + \beta)/2]$. Suppose that $X_m(z)$ is holomorphic in the interior of $\bar S_m$, continuous up to its boundary, and $\|X_m(z)\| \le 4^{-m}\, c\,|z|^{-2}$, which for $m = 0$ is correct. Then for any path of integration in $\bar S_m$, the above integral converges compactly for z not on the path. Consequently, $U_m(z)$ is holomorphic in interior points of $\bar S^*_m$ and tends to 0 as $z \to \infty$ in $\bar S^*_{m+1}$. Moreover, for $z \in \bar S^*_{m+1}$ we can always integrate along the boundary of $\bar S_m$, in which case $|u - z| \ge \delta_{m+1} - \delta_m = 2^{-m-1}\delta$. Therefore, $\|U_m(z)\| \le 2^{-m}\, Lc\,[\pi\,\delta]^{-1}$, $z \in \bar S^*_{m+1}$, where L is an upper bound for the integral of $|u|^{-2} d|z|$ along the corresponding path. By choosing $\tilde\rho$ large enough, we can ensure that $8\,L\,c \le \delta\,\pi$, so that $\|U_m(z)\| \le 2^{-m-3}$ in $\bar S^*_{m+1}$. Consequently, $I + U_m(z)$ is invertible in $\bar S^*_{m+1}$, and according to Exercise 1 on p. 104, its inverse can be bounded by 2. Therefore, we obtain $|z|^2 \|X_{m+1}(z)\| \le 2^{-3m-2} c \le 4^{-m-1} c$ in $\bar S_{m+1}$. Altogether, this shows that the above iterative definition works, producing sequences $U_m(z)$, $X_m(z)$ which are holomorphic in S^*_∞, resp. S_∞, being the intersection of S^*_m, resp. S_m. From the definition of $U_m(z)$, and using Cauchy's formula, we obtain $U_m(ze^{2\pi i}) - U_m(z) = X_m(z)$, $z \in S_\infty$, $m \in \mathbb{N}_0$. Using this and the definition of the $X_m(z)$, one then finds $[I + U_m(z)]\,[I + X_m(z)] = [I + X_{m+1}(z)]\,[I + U_m(ze^{2\pi i})]$. We now define matrices $T_m(z)$ by

$$I + T_m(z) = [I + U_m(z)] \cdot \ldots \cdot [I + U_0(z)], \quad m \ge 0,\ z \in S_\infty.$$

Owing to Exercise 4, the sequence $T_m(z)$ converges uniformly on S^*_∞, defining a matrix function $T(z)$ that is holomorphic there and vanishes for $z \to \infty$, since all the $T_m(z)$ do the same. According to the above relations for $U_m(z)$, we find $[I + T_m(z)]\,[I + X(z)] = [I + X_{m+1}(z)]\,[I + T_m(z\mathrm{e}^{2\pi i})]$. Since $X_m(z) \to 0$ for $m \to \infty$, uniformly on S_∞, this then implies $[I + T(z)]\,[I + X(z)] = I + T(z\mathrm{e}^{2\pi i})$ in S_∞, which completes the proof. □

In order to formulate in which sense the Stokes multipliers are free, consider any system (3.1) (p. 37). This system has an HLFFS $(\hat{F}(z), Y(z))$, and corresponding Stokes' multipliers V_j, which are restricted by (9.4) and the *support condition* stated in Theorem 45. Now, *consider any set of constant matrices $\tilde{V}_j$, $j \in \mathbb{Z}$, with $\tilde{V}_{j+j_0} = \mathrm{e}^{-2\pi i L}\tilde{V}_j\,\mathrm{e}^{2\pi i L}$, and $(n,m) \notin \mathrm{Supp}_j \Rightarrow \tilde{V}^{(j)}_{n,m} = 0$.* Using the above lemma, it is easy to prove existence of another system having an HLFFS with the same $Y(z)$ and these Stokes multipliers:

Theorem 49 *Under the assumptions stated above, there exists a system $z\,\tilde{x}' = \tilde{A}(z)\,\tilde{x}$ having an HLFFS $(I + \hat{T}(z)\,\hat{F}(z), Y(z))$, with a formal analytic transformation $I + \hat{T}(z)$ of Gevrey order k, and corresponding Stokes' multipliers $\tilde{V}_j$, $j \in \mathbb{Z}$.*

Proof: For the proof, it suffices to consider the case of $\tilde{V}_j = V_j$ for all but one index j between 0 and $j_0 - 1$; applying this result repeatedly, the general case follows immediately. For which index j we allow $\tilde{V}_j$ to differ from V_j is also inessential, because a change of variable $z \mapsto z\mathrm{e}^{id}$, for suitable d, may be used to shift the enumeration of the Stokes multipliers by any amount. Thus, for the proof of the above result we restrict ourselves to the case of $\tilde{V}_j = V_j$, $1 \le j \le j_0 - 1$, and we define W by $I + \tilde{V}_0 = (I + W)\,(I + V_0)$. According to Exercise 4 on p. 142, this matrix W also satisfies the support condition for $j = 0$, so that the $X(z) = X_0(z)\,W\,X_0^{-1}(z) \cong_{1/k} \hat{0}$ in $S_0 \cap S_{-1}$, whose boundary rays are $d_{-1} \pm \pi/(2k)$. Hence, Lemma 17 may be applied to this matrix $X(z)$ and any closed subsector $\bar{S}(\rho)$ of $S_0 \cap S_{-1}$. By choosing $\bar{S}(\rho)$ large, resp. $\delta > 0$ small enough, the matrix $T(z)$ so obtained is holomorphic in a corresponding sector S^* with boundary rays $d_{-1} - \pi/(2k) + \varepsilon$, resp. $d_{-1} + \pi(2 + 1/(2k)) - \varepsilon$, where $\varepsilon > 0$ is so small that no singular direction lies within $(d_{-1} - \varepsilon, d_{-1} + \varepsilon)$. Moreover, we may assume that $I + T(z)$ is invertible there; otherwise, the radius of S^* can be made larger. For $\tilde{X}(z) = [T + T(z)]\,X_0(z)$, one can verify $\tilde{X}(z\mathrm{e}^{2\pi i}) = \tilde{X}(z)\,[I + W]\,\exp[2\pi i\, M_0]$, with M_0 as in (9.5). *Defining* $\tilde{A}(z) = z\,\tilde{X}'(z)\,\tilde{X}^{-1}(z)$, we have a system for which $\tilde{X}(z)$ is a fundamental solution.

Since the matrix $X(z)$ is asymptotically zero of Gevrey order $1/k$, we conclude from Proposition 18 (p. 121) that $T(z) \cong_{1/k} \hat{T}(z)$ in S^*. Therefore, $[I + \hat{T}(z)]\,\hat{F}(z), Y(z))$ is an HLFFS of $z\,\tilde{x}' = \tilde{A}(z)\,\tilde{x}$, and its data pairs are the same as those of $(\hat{F}(z), Y(z))$. According to the *Main Theorem* in Section 8.3, $\hat{T}(z)$ is k-summable, with singular directions among those for the system (3.1). Owing to the size of S^*, we conclude that the k-sum of $\hat{T}(z)$ equals $T(z)$, for all directions d between $d_{-1} - \varepsilon$ and $d_{-1} + 2\pi + \varepsilon$. Because of the smallness of ε, this shows that d_1 is the only singular direction, modulo 2π, of $\hat{T}(z)$. Consequently, for $0 \le j \le j_0$, the matrices $\tilde{F}_j(z) =$

$[I+T(z)]\,F_j(z)$ are the HLNS corresponding to $[I+\hat{T}(z)]\,\hat{F}(z), Y(z))$. From this we obtain that the corresponding Stokes' multipliers $\tilde{V}_j$ equal V_j, for $1 \le j \le j_0$. The matrix $\tilde{F}_{-1}(z) = [I + T(z e^{2\pi i})]\,F_{-1}(z)$ has the correct asymptotic in S_{-1}, and hence is the corresponding HLNS. This implies $I + \tilde{V_0} = [I + W]\,(I + V_0)$, completing the proof. □

Exercises:

1. For $x \ge 0$, show $\log(1+x) \le x$.
2. For nonnegative real numbers x_m with $\sum_0^\infty x_m < \infty$, show that the sequence $p_n = \prod_{m=0}^n (1 + x_m)$ converges.
3. For arbitrary $\nu \times \nu$-matrices A_n, show $\|(I + A_m) \cdot \ldots \cdot (I + A_0) - I\| \le \prod_{n=0}^m (1 + \|A_n\|) - 1$.
4. Show that the sequence $T_m(z)$, defined in the proof of Lemma 17, converges uniformly on S_∞.

10
Multisummable Power Series

In previous chapters we have shown that HLFFS are very natural to consider when discussing Stokes' phenomenon, since they are k-summable, for some suitable $k > 0$. Unfortunately, their computation is not so easy, because of one step requiring use of *Banach's fixed point theorem*. So in applications one may prefer to work with *formal fundamental solutions* in the classical sense. They can be computed relatively easily, using computer algebra tools which will briefly be discussed in Section 13.5. In Section 8.4 we have shown these FFS to be a product of finitely many matrix power series, each of which is k-summable with a value of k depending on the factor. However, this factorization is neither unique nor fully constructive, so the problem of summation of FFS remains. In this chapter we are now presenting a summability method that is *stronger than k-summability* for every $k > 0$, enabling us to sum FFS as a whole. This method, named *multisummability*, was first introduced in somewhat different form by *Ecalle* [94], using what he called *acceleration operators*. Here, we present an equivalent definition, based on the more general integral operators introduced in Sections 5.5 and 5.6. We also show that it would be sufficient to work with Laplace operators only, but it can be convenient in applications to have the more general integral operators at hand: Sometimes one will meet formal power series for which it will be simpler to show applicability of some particular integral operator, but more complicated to do so for a Laplace operator, although theoretically they are equivalent.

As we stated above, multisummability will turn out to be stronger than k-summability, for every $k > 0$. On the other hand, it will be not too much stronger, as we are going to show that *every multisummable formal* *power series can be written as a sum of finitely many formal power series, each of which is k-summable, in some direction, for an individual $k > 0$, depending on the series.* Thus, roughly speaking, the set of multisummable power series is the linear hull of the union of the sets of k-summable ones. This description, although quite suggestive, is not entirely correct, as one also has to consider the corresponding directions, but that shall be made clearer later on.

Several authors have presented and/or applied Ecalle's theory of multisummability, e.g., *Martinet and Ramis* [186, 187], *Malgrange and Ramis* [185], *Loday-Richaud* [169, 170], *Thomann* [260–262], *Balser and Tovbis* [43], *Malgrange* [184], and *Jung, Naegele, and Thomann* [142].

10.1 Convolution Versus Iteration of Operators

In Sections 5.5 and 5.6 we introduced general kernel functions $e(z)$ and corresponding integral operators T. The most important examples of kernels of order $k > 0$ are $e(z) = k\, z^k \exp[z^k]$; the corresponding operators being Laplace resp. Borel operators. Other examples are Ecalle's acceleration operators, which shall be investigated in the following chapter.

Given two kernels $e_1(z)$, $e_2(z)$ of orders $k_1, k_2 > 0$ with corresponding operators T_1, T_2 and moment functions $m_1(u)$, $m_2(u)$, we have defined in Theorem 31 (p. 94) a new kernel $e(z)$ of order $k = (1/k_1 + 1/k_2)^{-1}$ with corresponding moment function $m_1(u)\, m_2(u)$. This kernel $e(z)$ shall be called the *convolution* $e_1 * e_2$ of e_1 and e_2. It defines an integral operator $T_1 * T_2$, for which we now investigate its relation with $T_1 \circ T_2$.

Lemma 18 *Under the above assumptions, let $f \in \boldsymbol{A}^{(k)}(S(d,\varepsilon),\mathbb{E})$, for some $d \in \mathbb{R}$, $\varepsilon > 0$, so that $T_1 * T_2\, f$ is defined. Then $g = T_2\, f$ is holomorphic in the sector $S(d, \varepsilon + \pi/k_2)$, and is of exponential growth not more than k_1 there; consequently, $T_1\, g$ is defined, integrating along any direction d_1 with $2|d - d_1| < \varepsilon + \pi/k_2$, and $T_1\, g = T_1 * T_2\, f$.*

Proof: Use the definition of $e_1 * e_2$ and justify interchanging the order of integration in the formula representing $T_1 * T_2\, f$. □

The above lemma shows that $T_1 * T_2$ equals $T_1 \circ T_2$ whenever the first one is defined. However, examples show that $T_1 \circ T_2$ can be applied to a wider set of functions. In a sense, this observation is the key to multisummability.

Exercises: In the following exercises, let kernels $e_j(z)$ of arbitrary orders $\kappa_j > 0$ $(1 \le j \le q)$ be given.

1. Show $e_1 * e_2 = e_2 * e_1$, $(e_1 * e_2) * e_3 = e_1 * (e_2 * e_3)$.

2. Let $f \in \boldsymbol{A}^{(k)}(S(d,\varepsilon),\mathbb{E})$, for $k^{-1} = \sum_1^q \kappa_j^{-1}$. Show $T_1 * \ldots * T_q\ f = T_1 \circ \ldots \circ T_q\ f$ and discuss the choice of directions of integration in

the multiple integral on the right-hand side. Furthermore, show that for f as above, $T_1 \circ \ldots \circ T_q$ does not depend on the enumeration of the T_j.

10.2 Multisummability in Directions

In what follows, we shall consider a fixed tuple $\boldsymbol{T} = (T_1, \ldots, T_q)$ of integral operators T_j of respective orders $\kappa_j > 0$. Furthermore, we consider a likewise fixed *multidirection* $d = (d_1, \ldots, d_q)$, which we call *admissible with respect to* $\boldsymbol{T}$, provided that

$$2\kappa_j\,|d_j - d_{j-1}| \le \pi, \quad 2 \le j \le q. \tag{10.1}$$

Given $\boldsymbol{T}$ and d, we are now going to define $\boldsymbol{T}$-summability in the multidirection d. To do so, we consider the *inverse of the formal operators* $\hat{T}_j$, i.e., $(\hat{T}_j^{-1}\,\hat{f})(z) = \sum f_n\, z^n/m_j(n)$, for $\hat{f}(z) = \sum f_n\, z^n \in \mathbb{E}\,[[z]]$.

- For $q = 1$, we say that a formal power series $\hat{f} \in \mathbb{E}\,[[z]]$ is $\boldsymbol{T}$-summable in the multidirection d, provided that $\hat{g} = \hat{T}_1^{-1}\hat{f} \in \mathbb{E}\,\{z\}$, and that its sum $g(z)$ is in $\boldsymbol{A}^{(\kappa_1)}(S(d_1,\varepsilon), \mathbb{E}\,)$, for some $\varepsilon > 0$.
- For $q \ge 2$, set $\tilde{\boldsymbol{T}} = (T_2, \ldots, T_q)$, $\tilde{d} = (d_2, \ldots, d_q)$. Then we say that a formal power series $\hat{f} \in \mathbb{E}\,[[z]]$ is $\boldsymbol{T}$-summable in the multidirection d, provided that $\hat{g} = \hat{T}_1^{-1}\hat{f}$ is $\tilde{\boldsymbol{T}}$-summable in the multidirection $\tilde{d}$, and that its $\tilde{\boldsymbol{T}}$-sum $g(z)$ is in $\boldsymbol{A}^{(\kappa_1)}(S(d_1,\varepsilon), \mathbb{E}\,)$, for some $\varepsilon > 0$.
- In both cases, note that the operator T_1 can be applied to the function $g(z)$, integrating along a ray inside the sector $S(d_1, \varepsilon)$. We then say that $f = T_1\, g$ is the $\boldsymbol{T}$-sum of $\hat{f}$ in the multidirection d, and write $f = \mathcal{S}_{\boldsymbol{T},d}\,\hat{f}$.

We shall write $\mathbb{E}\,\{z\}_{\boldsymbol{T},d}$ for the set of all formal power series that are $\boldsymbol{T}$-summable in the multidirection d. Observe for $q = 1$, that Theorem 38 (p. 108) implies $\mathbb{E}\,\{z\}_{\boldsymbol{T},d} = \mathbb{E}\,\{z\}_{\kappa_1,d}$.

Exercises: In what follows, let $\boldsymbol{T}$ and d, as above, be given.

1. Show that the above inductive definition of $\boldsymbol{T}$-summability in a multidirection d is equivalent to the following: Given $\hat{f}(z)$, we first divide the coefficients f_n by $m_1(n) \cdot \ldots \cdot m_q(n)$ to obtain a convergent series – hence $\hat{f}(z)$ has to be in $\mathbb{E}\,[[z]]_s$, with $s = 1/\kappa_1 + \ldots + 1/\kappa_q$, for this to hold. Then we apply the integral operator T_q, integrating in a direction close to d_q, then T_{q-1}, integrating close to d_{q-1}, and so forth.

The assumptions made are such that these operators all apply. In particular, the inequalities (10.1) assure that the function defined by application of the integral operator T_{j-1} is analytic for values of the variable close to the origin, having argument close to d_j. By assumption, this function then can be holomorphically continued to infinity and has the correct growth, so that the next operator applies.

2. Show that to $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$ there corresponds a tuple $(f_0(z), \ldots, f_q(z))$ of functions. The function $f_q(z)$ is the sum of the convergent series $\hat{f}_q = \hat{T}_q^{-1} \circ \ldots \circ \hat{T}_1^{-1} \hat{f}$. The others are successively obtained as $f_{j-1} = T_j f_j$, $1 \le j \le q$, integrating in a direction close to d_j. From Theorem 27 (p. 91), conclude

$$f_j(z) \cong_{s_j} \hat{f}_j(z) = (\hat{T}_j^{-1} \circ \ldots \circ \hat{T}_1^{-1} \hat{f})(z) \text{ in } G_j, \quad 0 \le j \le q-1,$$

where $s_j = \sum_{j+1}^{q} 1/\kappa_\nu$, and G_j is a sectorial region with bisecting direction d_{j+1} and opening larger than π/κ_{j+1}. The function f_0 then is the $\boldsymbol{T}$-sum of $\hat{f}$, and in particular $\hat{f}_0(z) = \hat{f}(z)$.

3. For $\hat{f}_1, \hat{f}_2 \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$, show that $\hat{f}_1 + \hat{f}_2 \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$, and we have $\mathcal{S}_{\boldsymbol{T},d}(\hat{f}_1 + \hat{f}_2) = \mathcal{S}_{\boldsymbol{T},d} \hat{f}_1 + \mathcal{S}_{\boldsymbol{T},d} \hat{f}_2$.

4. Let $e_j(z)$ be the kernel of T_j, and let $\hat{f} \in \mathbb{E}[[z]]$. For a natural number p, consider the operators $\tilde{T}_j$ with kernels $\tilde{e}_j(z) = p\, e_j(z^p)$ of corresponding orders $\tilde{\kappa}_j = p\kappa_j$, and set $\hat{g}(z) = \hat{f}(z^p)$. Show that $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$ if and only if $\hat{g} \in \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}}$, with $\tilde{d} = p^{-1} d$. If this is so, show $(\mathcal{S}_{\boldsymbol{T},d} \hat{f})(z) = (\mathcal{S}_{\tilde{\boldsymbol{T}},\tilde{d}}\, \hat{g})(z^{1/p})$, wherever both sides are defined.

5. For $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$, define f_j and $\hat{f}_j$ as in Exercise 2. For each j, $1 \le j \le q-1$, show: $\hat{f}_j \in \mathbb{E}\{z\}_{\boldsymbol{T}(j),d(j)}$, with $\boldsymbol{T}(j) = (T_{j+1}, \ldots, T_q)$, $d(j) = (d_{j+1}, \ldots, d_q)$, and $\mathcal{S}_{\boldsymbol{T}(j),d(j)} \hat{f}_j = f_j$.

6. For $k = (1/\kappa_1 + \ldots + 1/\kappa_q)^{-1}$, show $\mathbb{E}\{z\}_{k,d_q} \subset \mathbb{E}\{z\}_{\boldsymbol{T},d}$, and $\mathcal{S}_{\boldsymbol{T},d} \hat{f} = \mathcal{S}_{k,d_q} \hat{f}$ for every $\hat{f} \in \mathbb{E}\{z\}_{k,d_q}$.

10.3 Elementary Properties

The following lemma lists properties of multisummability that are direct consequences of the definition, and the proof can be left to the reader:

Lemma 19

(a) *Let* $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$ *and* j, $1 \le j \le q$, *be given. Then there exists* $\varepsilon > 0$ *so that* $f \in \mathbb{E}\{z\}_{\boldsymbol{T},\tilde{d}}$ *and* $\mathcal{S}_{\boldsymbol{T},\tilde{d}} \hat{f} = \mathcal{S}_{\boldsymbol{T},d} \hat{f}$ *hold for every* $\tilde{d} = (d_1, \ldots, d_{j-1}, \tilde{d}_j, d_{j+1}, \ldots, d_q)$ *satisfying* $|\tilde{d}_j - d_j| < \varepsilon$, $2\kappa_j |\tilde{d}_j - d_{j-1}| \le \pi$, *and* $2\kappa_{j+1} |d_{j+1} - \tilde{d}_j| \le \pi$.

(b) *Let* $\hat{f} \in \mathbb{E}[[z]]$ *be given, and let* $d = (d_1, \ldots, d_q)$ *be admissible with respect to* $\boldsymbol{T}$. *Then* $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_q)$ *with* $\tilde{d}_j = d_j + 2\pi$, $1 \le j \le n$, *is also admissible with respect to* $\boldsymbol{T}$, *and* $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$ *if and only if* $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},\tilde{d}}$. *If so, then* $(\mathcal{S}_{\boldsymbol{T},d} \hat{f})(z) = (\mathcal{S}_{\boldsymbol{T},\tilde{d}} \hat{f})(z e^{2\pi i})$, *for every* z *where either side is defined.*

Let $\boldsymbol{T} = (T_1, \ldots, T_q)$ and $d = (d_1, \ldots, d_q)$, admissible with respect to $\boldsymbol{T}$, be given, and assume $q \ge 2$. With $1 \le \nu \le q$, define $\tilde{\boldsymbol{T}} = (T_1, \ldots, T_{q-1})$ in case $\nu = q$, resp. $= (T_1, \ldots, T_{\nu-1}, T_\nu * T_{\nu+1}, T_{\nu+2}, \ldots, T_q)$ otherwise, and $\tilde{d} = (d_1, \ldots, d_{\nu-1}, d_{\nu+1}, \ldots, d_q)$. It is easily seen that then $\tilde{d}$ is admissible with respect to $\tilde{\boldsymbol{T}}$.

Lemma 20 *For* $q \ge 2$ *and* $\boldsymbol{T}$, d, *resp.* $\tilde{\boldsymbol{T}}$, $\tilde{d}$ *as above, we have* $\mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}} \subset \mathbb{E}\{z\}_{\boldsymbol{T},d}$, *and for every* $\hat{f} \in \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}}$ *we have* $(\mathcal{S}_{\tilde{\boldsymbol{T}},\tilde{d}} \hat{f})(z) = (\mathcal{S}_{\boldsymbol{T},d} \hat{f})(z)$ *wherever both sides are defined, which holds at least for* z *with* $|z|$ *sufficiently small and* $|d_1 - \arg z| \le \pi/(2\kappa_1)$.

Proof: Let $\hat{f} \in \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}}$ be given. In case $\nu = q$, we find that $g = \mathcal{S} \circ \hat{T}_q^{-1} \circ \ldots \circ \hat{T}_1^{-1} \hat{f}$ is entire and of exponential growth at most κ_q in every sector of infinite radius. Hence we conclude that $\tilde{g} = T_q\, g = \mathcal{S} \circ T_{q-1}^{-1} \circ \ldots \circ \hat{T}_1^{-1} \hat{f}$, where integration in the integral operator may be along any ray. This completes the proof in this case. In case $1 \le \nu \le q-1$, let g be as above, and define $\tilde{g} = T_{\nu+2} \circ \ldots \circ T_q\, g$; in particular, let $\tilde{g} = g$ for $\nu = q-1$. Then from Lemma 18 (p. 160) we find that $T_\nu * T_{\nu+1}\, \tilde{g} = T_\nu \circ T_{\nu+1}\, \tilde{g}$, and this is all we need to show. □

Exercises: For $\boldsymbol{T}$, d, $\tilde{\boldsymbol{T}}$, $\tilde{d}$ as above, we define the *projection* π_ν by $\pi_\nu(\boldsymbol{T}, d) = (\tilde{\boldsymbol{T}}, \tilde{d})$.

1. For $1 \le \nu_1 < \ldots < \nu_p \le q$ $(1 \le p \le q-1)$, let $(\tilde{\boldsymbol{T}}, \tilde{d}) = \pi_{\nu_1} \circ \ldots \circ \pi_{\nu_p}(\boldsymbol{T}, d)$. Show $\mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}} \subset \mathbb{E}\{z\}_{\boldsymbol{T},d}$. Moreover, show for every $\hat{f} \in \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}}$ that $(\mathcal{S}_{\tilde{\boldsymbol{T}},\tilde{d}} \hat{f})(z) = (\mathcal{S}_{\boldsymbol{T},d} \hat{f})(z)$ wherever both sides are defined.

2. Show $\mathbb{E}\{z\}_{k_j,d_j} \subset \mathbb{E}\{z\}_{\boldsymbol{T},d}$, for k_j defined by $1/k_j = \sum_1^j 1/\kappa_\ell$, $1 \le j \le q$. Moreover, for every $\hat{f} \in \mathbb{E}\{z\}_{k_j,d_j}$ show $(\mathcal{S}_{k_j,d_j} \hat{f})(z) = (\mathcal{S}_{\boldsymbol{T},d} \hat{f})(z)$ wherever both sides are defined.

3. Assume $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d} \cap \mathbb{E}[[z]]_{1/k_{q-1}}$. Show that then $\hat{f} \in \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}}$, for $(\tilde{\boldsymbol{T}}, \tilde{d}) = \pi_q(\boldsymbol{T}, d)$.

10.4 The Main Decomposition Result

Let integral operators $\boldsymbol{T} = (T_1, \ldots, T_q)$ of respective orders $\kappa_1, \ldots, \kappa_q$, and a multidirection $d = (d_1, \ldots, d_q)$, admissible with respect to $\boldsymbol{T}$, be given, and define $k = (k_1, \ldots, k_q)$ as in Exercise 2 on p. 163. It follows from this and Exercise 3 on p. 162 that $\hat{f}_j \in \mathbb{E}\{z\}_{k_j,d_j}$, $1 \le j \le q$, implies $\hat{f} = \sum \hat{f}_j \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$. In this section we will show in case of $\kappa_j > 1/2$, $1 \le j \le q$, that every $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$ is obtained in this fashion. To do so, we use the following lemma:

Lemma 21 *Let real numbers $\tilde{k} > k > 0$ be given, so that $\tilde{k} > 1/2$, and define κ by $1/\kappa = 1/k - 1/\tilde{k}$. Let T be any integral operator of order κ. For $\hat{h} \in \mathbb{E}\{z\}_{\tilde{k},\tilde{d}}$, for some real number $\tilde{d}$, define $\hat{f} = \hat{T}\,\hat{h}$. Then $\hat{f} = \hat{f}_1 + \hat{f}_2$, where $\hat{f}_1 \in \mathbb{E}[[z]]_{1/\kappa}$ and $\hat{f}_2 \in \mathbb{E}\{z\}_{k,\tilde{d}}$.*

Proof: By definition, $h = \mathcal{S}_{\tilde{k},\tilde{d}}\,\hat{h}$ is holomorphic in a sectorial region $G = G(\tilde{d}, \alpha)$, with $\alpha > \pi/\tilde{k}$, and $h(z) \cong_{1/\tilde{k}} \hat{h}(z)$ in G. Let $\bar{S} = \bar{S}(\tilde{d}, \tilde{\alpha}, \tilde{\rho})$ be a closed subsector of G with $2\pi > \tilde{\alpha} > \pi/\tilde{k}$, and let γ denote the positively oriented boundary of $\bar{S}$. Decomposing $\gamma = \gamma_1 + \gamma_2$, where γ_1 is the circular part of γ, let

$$h_j(z) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{h(w)}{w - z}\, dw, \quad z \in S(\tilde{d}, \tilde{\alpha}, \tilde{\rho}),\ j = 1, 2.$$

Then $h = h_1 + h_2$ by Cauchy's Formula, and h_1 is holomorphic at the origin. Therefore, $h_2(z) = h(z) - h_1(z) \cong_{1/\tilde{k}} \hat{h}_2(z)$ in $S(\tilde{d}, \tilde{\alpha}, \tilde{\rho})$, and $\hat{h}_2(z) = \hat{h}(z) - \hat{h}_1(z)$, where $\hat{h}_1(z)$ is the power series of $h_1(z)$, and hence converges. So $\hat{f}_1 = \hat{T}\,\hat{h}_1 \in \mathbb{E}[[z]]_{1/\kappa}$. Moreover, h_2 remains holomorphic in $\hat{S} = \hat{S}(\tilde{d}, \tilde{\alpha})$, tending to zero as $z \to \infty$ in $\hat{S}$. So Theorem 27 (p. 91) implies $f_2(z) = (T\,h_2)(z) \cong_{1/k} \hat{f}_2(z) = (\hat{T}\,\hat{h}_2)(z)$ in a sector of opening larger than π/k and bisecting direction $\tilde{d}$. Hence by definition $\hat{f}_2 \in \mathbb{E}\{z\}_{k,\tilde{d}}$. □

We are now ready to prove the main decomposition result [17, 19]:

Theorem 50 (Main Decomposition Theorem) *Let $\boldsymbol{T} = (T_1, \ldots, T_q)$ and $d = (d_1, \ldots, d_q)$, admissible with respect to $\boldsymbol{T}$, be given, and define k_j by $1/k_j = \sum_1^j 1/\kappa_\ell$, $1 \le j \le q$. If all $\kappa_j > 1/2$, for $1 \le j \le q$, then for $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$ we have $\hat{f} = \sum_{j=1}^q \hat{f}_j$, with $\hat{f}_j \in \mathbb{E}\{z\}_{k_j,d_j}$, $1 \le j \le q$, and $\mathcal{S}_{\boldsymbol{T},d}\,\hat{f} = \sum_{j=1}^q \mathcal{S}_{k_j,d_j}\,\hat{f}_j$.*

Proof: We proceed by induction with respect to q: In case $q = 1$, the statement holds trivially, hence assume $q \geq 2$. From Exercise 5 on p. 162 we conclude that $\hat{h}(z) = \hat{T}_{q-1}^{-1} \circ \ldots \circ \hat{T}_1^{-1} \hat{f} \in \mathbb{E}\{z\}_{\kappa_q, d_q}$. Set $k = k_q$, $\tilde{k} = \kappa_q$, then the number κ in Lemma 21 equals k_{q-1}. Applying the lemma with $\tilde{d} = d_q$ and $T = T_1 * \ldots * T_{q-1}$, we find $\hat{f} = \hat{f}_1 + \hat{f}_2$, with $\hat{f}_2 \in \mathbb{E}\{z\}_{k,d_q} \subset \mathbb{E}\{z\}_{\boldsymbol{T},d}$, and $\hat{f}_1 \in \mathbb{E}[[z]]_{1/k_{q-1}}$. Since $\hat{f}_2 \in \mathbb{E}\{z\}_{\boldsymbol{T},d}$, we have $\hat{f}_1 \in \mathbb{E}\{z\}_{\boldsymbol{T},d} \cap \mathbb{E}[[z]]_{1/k_{q-1}}$, hence according to Exercise 3 on p. 164 we find $\hat{f}_1 \in \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},\tilde{d}}$, with $\tilde{\boldsymbol{T}} = (T_1, \ldots, T_{q-1})$, $\tilde{d} = (d_1, \ldots, d_{q-1})$. Applying the induction hypothesis to $\hat{f}_1$, we see that the proof is completed. □

The last theorem has the following important consequence:

Corollary to Theorem 50 *Suppose that we have tuples $\boldsymbol{T} = (T_1, \ldots, T_q)$ and $\tilde{\boldsymbol{T}} = (\tilde{T}_1, \ldots, \tilde{T}_q)$ of integral operators with T_j and $\tilde{T}_j$ having the same order $\kappa_j > 0$, for $1 \leq j \leq q$. Then $\mathbb{E}\{z\}_{\boldsymbol{T},d} = \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},d}$ for every multidirection d satisfying* (10.1)*, and $\mathcal{S}_{\boldsymbol{T},d} \hat{f} = \mathcal{S}_{\tilde{\boldsymbol{T}},d} \hat{f}$ for every $\hat{f} \in \mathbb{E}\{z\}_{\boldsymbol{T},d} = \mathbb{E}\{z\}_{\tilde{\boldsymbol{T}},d}$.*

Proof: Use Exercise 4 on p. 162 to see that we may assume $\kappa_j > 1/2$, in which case the assertion follows directly from Theorem 50, Exercise 2 on p. 163 and Exercise 3 on p. 162. □

In view of the above corollary, we shall from now on use the following terminology:

- A tuple $k = (k_1, \ldots, k_q)$ will be called *a type of multisummability*, or simply, *admissible*, provided that $k_1 > k_2 > \ldots > k_q > 0$. Given such a k, we always define $\kappa_1, \ldots, \kappa_q$ by $\kappa_1 = k_1$, $1/\kappa_j = 1/k_j - 1/k_{j-1}$, $2 \leq j \leq q$, so that in turn k_j are as in Theorem 50.

- Given a type k of multisummability, we shall say that $\hat{f}(z)$ is k-summable in an admissible multidirection d, if there exists a tuple $\boldsymbol{T} = (T_1, \ldots, T_q)$ of integral operators of respective orders $\kappa_1, \ldots, \kappa_q$ so that $\hat{f}(z)$ is $\boldsymbol{T}$-summable in the multidirection d. From the corollary to Theorem 50, we then conclude that the choice of the operators T_j is completely arbitrary, provided that they have the required orders. In particular, the sum $\mathcal{S}_{\boldsymbol{T},d} \hat{f}$ only depends on the multisummability type k and the multidirection d; hence from now on we shall always write $\mathcal{S}_{k,d} \hat{f}$ and $\mathbb{E}\{z\}_{k,d}$ instead of $\mathcal{S}_{\boldsymbol{T},d} \hat{f}$ and $\mathbb{E}\{z\}_{\boldsymbol{T},d}$.

Instead of the k_j, we could also use the orders κ_j of the integral operators to define the type of multisummability, but in the literature the use of k_j is more common, because for formal solutions of systems (3.1) (p. 37) they agree with the set of levels defined in Section 8.4.

The above results show that in the theory of multisummability one can arbitrarily choose the operators $T_1, \ldots, T_q$ having the given orders.

For practical purposes, one therefore may always take Laplace operators $\mathcal{L}_{\kappa_1}, \ldots, \mathcal{L}_{\kappa_q}$. This case has been studied in [18] under the name of *summation by iterated Laplace integrals*. For more theoretical purposes, the original definition of Ecalle in terms of *acceleration operators* sometimes is more appropriate, since it has very natural properties as far as convolution of power series is concerned.

Exercises: In view of Exercise 4 on p. 162, we will extend the definition of multisummability to power series in a root $z^{1/p}$, $p \geq 2$ as follows: A power series $\hat{f}$ in $z^{1/p}$ is called k-summable in the multidirection d if and only if $\hat{f}(z^p)$ is pk-summable in the multidirection d/p.

1. Given admissible k and d, and $\hat{f} \in \mathbb{E}\{z\}_{k,d}$, show for sufficiently large natural p (depending only on k) that $\hat{f} = \sum_{j=1}^{q} \hat{f}_j$, with formal power series $\hat{f}_j$ in $z^{1/p}$ which are k_j-summable in direction d_j, for $1 \leq j \leq q$.
2. For $k_1 > k_2 > 0$ with $1/\kappa = 1/k_2 - 1/k_1 \geq 2$, let $|d_2 - d_1| \leq \pi/(2\kappa)$, so that $d = (d_1, d_2)$ is admissible with respect to $k = (k_1, k_2)$. For $\hat{f}_j \in \mathbb{E}\{z\}_{k_j,d_j}$, $j = 1, 2$, conclude $\hat{f} = \hat{f}_1 + \hat{f}_2 \in \mathbb{E}\{z\}_{k,d}$. Show that then $\hat{g} = \hat{\mathcal{B}}_{k_1} \hat{f}$ is κ-summable in direction d_2, hence $g = \mathcal{S}_{\kappa,d_2}\, \hat{g}$ is holomorphic in a sector $S = S(d_2, \alpha, r)$ of opening larger than 2π. Moreover, show that $\psi(z) = g(z) - g(z e^{2\pi i})$ can be holomorphically continued into a sector of infinite radius and bisecting direction $d_2 - \pi$.
3. For k_1, k_2 as in Ex. 3, show the existence of $\hat{f} \in \mathbb{E}\{z\}_{k,d}$, $k = (k_1, k_2)$, $d = (d_1, d_2)$, which *cannot* be written as $\hat{f} = \hat{f}_1 + \hat{f}_2$, $\hat{f}_j \in \mathbb{E}\{z\}_{k_j,d_j}$, $j = 1, 2$.
4. Under the assumptions of Theorem 50 (p. 164), show that the decomposition of $\hat{f}$ into a sum $\sum_{j=1}^{q} \hat{f}_j$, in case $q \geq 2$, is never unique.

10.5 Some Rules for Multisummable Power Series

The following is the analogue to some theorems in Section 6.3. To generalize the remaining ones will be easier using a result that we shall derive in Section 10.7.

Theorem 51 *For every admissible k and d, we have the following:*

(a) *If $\hat{f}, \hat{g}_1, \hat{g}_2 \in \mathbb{E}\{z\}_{k,d}$, then we have:*

$$\hat{g}_1 + \hat{g}_2 \in \mathbb{E}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{g}_1 + \hat{g}_2) = \mathcal{S}_{k,d}\, \hat{g}_1 + \mathcal{S}_{k,d}\, \hat{g}_2,$$

$$\hat{f}' \in \mathbb{E}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{f}') = \frac{d}{dz}(\mathcal{S}_{k,d}\, \hat{f}),$$

$$\int_0^z \hat{f}(w)dw \in \mathbb{E}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\left(\int_0^z \hat{f}(w)dw\right) = \int_0^z (\mathcal{S}_{k,d}\, \hat{f})(w)\, dw.$$

(b) *If* $\hat f \in \mathbb{E}\{z\}_{k,d}$ *and* p *is a natural number, then* $\hat g(z) = \hat f(z^p) \in \mathbb{E}\{z\}_{pk,p^{-1}d}$, *and* $(\mathcal{S}_{pk,p^{-1}d}\,\hat g)(z) = (\mathcal{S}_{k,d}\,\hat f)(z^p)$.

(c) *If* $\hat f \in \mathbb{E}\{z\}_{k,d}$ *and* p *is a natural number for which* $\hat g(z) = \hat f(z^{1/p})$ *is again a power series, then* $\hat g(z) \in \mathbb{E}\{z\}_{p^{-1}k,pd}$, *and* $(\mathcal{S}_{p^{-1}k,pd}\,\hat g)(z) = (\mathcal{S}_{k,d}\,\hat f)(z^{1/p})$.

Proof: Statements (b) and (c) follow directly from the definition and Exercise 2 on p. 88. For (a), we may assume that the numbers κ_j all are larger than 1/2, because if not we may use (b), (c) with sufficiently large p. In this case, (a) follows from Theorem 50 (p. 164) and the corresponding results in Section 6.3. □

While some of the rules for k-summability have been generalized to multisummability, we did not yet do so for those concerning *products* of series. This shall be done later with help of a result that characterizes those functions arising as sums of multisummable series.

Exercises: Let admissible k and d be given.

1. Show that the exercises in Section 6.3 generalize to multisummability.
2. Let $\hat f(z) = \sum_{n=-p}^{\infty} f_n\, z^n$ be a formal Laurent series, with $p \in \mathbb{N}$ and $f_n \in \mathbb{E}$. There are two ways of defining multisummability of $\hat f$:

   (a) We say that $\hat f$ is k-summable in the multidirection d if $\hat g(z) = \sum_{n=0}^{\infty} f_n\, z^n \in \mathbb{E}\{z\}_{k,d}$, and we then define $\mathcal{S}_{k,d}\,\hat f = \mathcal{S}_{k,d}\,\hat g + \sum_{n=-p}^{-1} f_n z^n$.

   (b) We say that $\hat f$ is k-summable in the multidirection d if $\hat g(z) = z^p\,\hat f(z) \in \mathbb{E}\{z\}_{k,d}$, and we then define $\mathcal{S}_{k,d}\,\hat f = z^{-p}\mathcal{S}_{k,d}\,\hat g$.

   Show that both definitions are equivalent, and in case $f_{-p} = \ldots = f_{-1} = 0$ coincide with the original definition for power series.

10.6 Singular Multidirections

Let a multisummability type $k = (k_1, \ldots, k_q)$ and a formal power series $\hat f(z) \in \mathbb{E}[[z]]$ be given. A multidirection d, admissible with respect to k, will be called *singular* for $\hat f$, if and only if $\hat f \notin \mathbb{E}\{z\}_{k,d}$; otherwise, we say that d is *nonsingular*. It is possible that *all* d are singular, e.g., when $\hat f \notin \mathbb{E}[[z]]_{1/k_q}$, or when $g = \mathcal{S}\,(\hat{\mathcal{B}}_{k_q}\hat f)$ cannot be holomorphically continued across the boundary of some bounded region containing the origin. The set of all singular multidirections will be called *the singular set of* $\hat{f}$ (with respect to k).

Inspecting the definition of $\mathbb{E}\{z\}_{k,d}$, one sees that the reason for $\hat{f} \notin \mathbb{E}\{z\}_{k,d}$, i.e., d singular, will be connected to the "level" j, since it may be so that the functions $f_j, \ldots, f_q$, defined in Exercise 2 on p. 162, all exist, but f_j cannot be holomorphically continued into any sector of infinite radius and bisecting direction d_j, or in every such sector has exponential growth larger than κ_j, so that application of the next operator fails. If this occurs, we shall say that *d is singular of level j.*

Let $d = (d_1, \ldots, d_q)$ be singular of level j for $\hat{f}$. Then the above discussion implies that all multidirections $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_q)$ with $\tilde{d}_\nu = d_\nu$ for $j \le \nu \le q$ are automatically singular of level j for $\hat{f}$, and we shall *identify all these singular multidirections.* Moreover, in view of Lemma 19 (p. 163) we may also identify admissible multidirections d and $\tilde{d}$ with $d_j - \tilde{d}_j = 2\mu\pi$, for some $\mu \in \mathbb{Z}$. After doing so, the singular set of $\hat{f}$ may or may not contain finitely many elements. If it does, we shall say that *$\hat{f}$ is k-summable.* As for $q = 1$ we shall write $\mathbb{E}\{z\}_k$ for the set of all k-summable formal power series with coefficients in $\mathbb{E}$.

For $q = 1$ we have seen that absence of singular directions implies convergence of the series $\hat{f}$. This generalizes to arbitrary q as follows:

Proposition 22 *Let $k = (k_1, \ldots, k_q)$, $q \ge 2$, be admissible, and assume that $\hat{f} \in \mathbb{E}\{z\}_k$ has no singular multidirections of level j, for some fixed j, $1 \le j \le q$. Then $\hat{f} \in \mathbb{E}\{z\}_{\tilde{k}}$, with $\tilde{k} = \pi_j(k) = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_q)$. Moreover, for every nonsingular multidirection $d = (d_1, \ldots, d_q)$ corresponding to k, the multidirection $\tilde{d} = (d_1, \ldots, d_{j-1}, d_{j+1}, \ldots, d_q)$ is nonsingular with respect to $\tilde{k}$, and $\mathcal{S}_{k,d}\,\hat{f} = \mathcal{S}_{\tilde{k},\tilde{d}}\,\hat{f}$.*

Proof: Without loss of generality, we assume that the operators T_j used are all Laplace operators of orders κ_j. Absence of singular multidirections of level $j = q$ implies that $f_q = \mathcal{S}\,(\hat{T}_q^{-1} \circ \ldots \circ \hat{T}_1^{-1}\,\hat{f})$ is entire and of exponential growth κ_q in arbitrary sectors of infinite radius. This shows that $f_{q-1} = T_q\, f_q$ is holomorphic at the origin, completing the proof in this case. For $j \le q-1$, consider any nonsingular multidirection d and the corresponding functions $f_0, \ldots, f_q$ defined in Exercise 2 on p. 162. Absence of singular multidirections of this level shows existence of a sector S of infinite radius, bisecting direction d_{j+1} and opening more than π/κ_{j+1} in which f_j is holomorphic and of exponential growth at most κ_j. For $f_{j+1} = T_{j+1}^{-1}\, f_j$, we obtain from Exercise 2 on p. 82[1] that $f_{j+1}(z)$ is of exponential growth not larger than k, with $1/k = 1/\kappa_j + 1/\kappa_{j+1}$, in a small sector with bisecting direction d_{j+1}. Consequently, the operator $T_j * T_{j+1}$ can be applied to connect f_{j+1} with f_{j-1}. This is equivalent to saying that

[1]By choice of the operators, T_{j+1}^{-1} is the Borel operator of order κ_{j+1}.

$\hat{f}$ is $\tilde{k}$-summable in the multidirection $\tilde{d} = (d_1, \ldots, d_{j-1}, d_{j+1}, \ldots, d_q)$, and $f_0 = \mathcal{S}_{\tilde{k},\tilde{d}}\,\hat{f}$. □

This result has the following converse:

Proposition 23 *Let $k = (k_1, \ldots, k_q)$, $q \geq 2$, be admissible, and set $\tilde{k} = \pi_j(k) = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_q)$ for some fixed j, $1 \leq j \leq q$. Then every $\hat{f} \in \mathbb{E}\{z\}_{\tilde{k}}$ is also in $\mathbb{E}\{z\}_k$, and therefore $\hat{f}$ has no singular multidirections of level j. In other words, a multidirection $d = (d_1, \ldots, d_q)$ is nonsingular for $\hat{f}$, regarded as an element of $\mathbb{E}\{z\}_k$, if and only if $\tilde{d} = (d_1, \ldots, d_{j-1}, d_{j+1}, \ldots, d_q)$ is nonsingular for $\hat{f}$ in $\mathbb{E}\{z\}_{\tilde{k}}$.*

Proof: The proof is obvious for $j = q$, so assume otherwise. For operators $T_1, \ldots, T_q$ of respective orders $\kappa_1, \ldots, \kappa_q$, the collection

$$T_1, \ldots, T_{j-1}, T_j * T_{j+1}, T_{j+2}, \ldots, T_q$$

can serve as operators for $\tilde{k}$-summability. Using Lemma 18 (p. 160), one can then replace $T_j * T_{j+1}$ by the iteration $T_j \circ T_{j+1}$, choosing any admissible direction to integrate along. This then completes the proof. □

Exercises: In the following exercises, consider some admissible $k = (k_1, \ldots, q_q)$ with $q \geq 2$.

1. Let $\hat{f} \in \mathbb{E}\{z\}_k \cap \mathbb{E}[[z]]_{1/\kappa}$, for $\kappa > k_q$. Show $\hat{f} \in \mathbb{E}\{z\}_{\tilde{k}}$, $\tilde{k} = \pi_q(k)$.

2. Let $\hat{f} \in \mathbb{E}\{z\}_k$ have no singular multidirections of whatever level. Show that then $\hat{f}$ converges.

10.7 Applications of Cauchy-Heine Transforms

The following two propositions characterize functions f that are the sum of multisummable power series. To do so, we define for every admissible $k = (k_1, \ldots, k_q)$ and $d = (d_1, \ldots, d_q)$ *closed intervals* $I_1, \ldots, I_q$ *corresponding to* d by $I_j = [d_j - \pi/(2k_j),\ \ d_j + \pi/(2k_j)]$, $1 \leq j \leq q$. Admissibility of d with respect to k then is equivalent to the inclusions $I_1 \subset I_2 \subset \ldots \subset I_q$. We also set $k_0 = \infty$ and recall that $f(z) \cong_0 \hat{f}(z)$ in S implies that $\hat{f}$ will converge and be the power series expansion of f about the origin. In particular, if $\hat{f} = \hat{0}$, then f vanishes identically.

Proposition 24 *Let $k = (k_1, \ldots, k_q)$ and $d = (d_1, \ldots, d_q)$ be admissible, let $p \in \mathbb{N}$ with $p\,k_q > 1/2$, and let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$. Then we can find $\varepsilon, \rho > 0$ such that to every $\phi \in \mathbb{R}$ there exists a function $f(z;\phi)$, holomorphic in $S_\phi = S(\phi, \varepsilon, \rho)$, such that the following holds:*

(a) *$f(z;\phi) \cong_{1/k_q} \hat f(z)$ in S_ϕ, for every $\phi \in \mathbb{R}$.*

(b) *For every ϕ_1, ϕ_2 with $|\phi_1 - \phi_2| < \varepsilon$, i.e., $S_{\phi_1} \cap S_{\phi_2} \neq \emptyset$, and every $j = 1, \dots, q$ we have the following: If $\phi_1, \phi_2 \in I_j$, then $f(z;\phi_1) - f(z;\phi_2) \cong_{1/k_{j-1}} \hat 0$ in $S_{\phi_1} \cap S_{\phi_2}$.*

(c) *$f(z;\phi) = f(z\mathrm{e}^{2p\pi i};\ \phi + 2p\pi)$ in S_ϕ, for every $\phi \in \mathbb{R}$.*

Proof: Using Theorem 51, we see that we may, without loss of generality, restrict ourselves to cases with $k_q > 1/2$, so that we may take $p = 1$. In this situation, Theorem 50 implies $\hat f = \sum_{j=1}^q \hat f_j$, with $\hat f_j \in \mathbb{E}\{z\}_{k_j,d_j}$. Hence $f(z) = (\mathcal{S}_{k,d}\,\hat f)(z) = \sum_{j=1}^q f_j(z)$, with $f_j = \mathcal{S}_{k_j,d_j}\,\hat f_j$ holomorphic and asymptotic to $\hat f_j$ of Gevrey order k_j in $\tilde S_j = S(d_j, \alpha_j, \rho)$, $\rho > 0$, $\alpha_j > \pi/k_j$. Taking $\varepsilon > 0$ so small that $\phi \in I_j$ implies $S_\phi \subset \tilde S_j$, we define $f(z;\phi) = \sum_{j=1}^q f_j(z;\phi)$, with $f_j(z;\phi) = f_j(z)$ if $\phi \in I_j$, while for $\phi \notin I_j$ we take *any* $f_j(z;\phi) \cong_{1/k_j} \hat f_j(z)$ in S_ϕ. Since ϕ and $\phi + 2\pi$ cannot both be in I_j, we can in particular arrange $f_j(z;\phi) = f_j(ze^{2\pi i};\phi + 2\pi)$. Using $k_1 > \ldots > k_q$ and $I_1 \subset I_2 \subset \ldots \subset I_q$, one can check that (a)–(c) hold. □

The next result is, in a sense, the converse of the above one:

Proposition 25 *Let $k = (k_1, \dots, k_q)$ and $d = (d_1, \dots, d_q)$ be admissible, with $k_q > 1/2$, and let $I_0 = [d_q - \pi, d_q + \pi]$. For $\varepsilon, \rho > 0$ and $\phi \in I_0$, assume existence of $f(z;\phi)$, holomorphic in $S_\phi = S(\phi, \varepsilon, \rho)$, such that the following holds:*

(a) *$f(z;\phi)$, in the variable z, is bounded at the origin, for every $\phi \in I_0$.*

(b) *For every $\phi_1, \phi_2 \in I_0$ with $|\phi_1 - \phi_2| < \varepsilon$ we have the following: If $\phi_1, \phi_2 \in I_j$ for some j, $1 \le j \le q$, then $f(z;\phi_1) - f(z;\phi_2) \cong_{1/k_{j-1}} \hat 0$ in $S_{\phi_1} \cap S_{\phi_2}$. If ϕ_1 or ϕ_2 is not in I_q, then $f(z;\phi_1) - f(z;\phi_2) \cong_{1/k_q} \hat 0$ in $S_{\phi_1} \cap S_{\phi_2}$.*

(c) *$f(z; d_q - \pi) = f(ze^{2\pi i}; d_q + \pi)$ in S_{ϕ_0}.*

Then there exists a unique $\hat f \in \mathbb{E}\{z\}_{k,d}$ with $f(z;d_1) = (\mathcal{S}_{k,d}\,\hat f)(z)$ in S_{d_1}.

Proof: Take any partitioning $\phi_0 = d_q - \pi < \phi_1 < \ldots < \phi_{m-1} < \phi_m = \phi_0 + 2\pi$ of the interval I_0 with $\phi_j - \phi_{j-1} < \varepsilon$, $1 \le j \le m$, and such that all the boundary points of all intervals I_ν occur in the set of ϕ_j. For $a_j = \tilde\rho \exp[i(\phi_j + \phi_{j-1})/2]$, with $0 < \tilde\rho < \rho$, define

$$g_j = \mathcal{CH}_{a_j}\left(f(\cdot;\phi_j) - f(\cdot;\phi_{j-1})\right), \quad 1 \le j \le m.$$

Then $g_j(z) \equiv 0$ whenever $\phi_j, \phi_{j-1} \in I_1$. Otherwise, let ν be taken maximally so that ϕ_j or ϕ_{j-1} is not in I_ν, i.e., $\phi_j \le d_\nu - \pi/(2k_\nu)$ or $\phi_{j-1} \ge d_\nu + \pi/(2k_\nu)$. Then Proposition 17 (p. 116) implies $g_j(z) \cong_{1/k_\nu} \hat g_j(z)$ in

$S(\tilde{d}_j, \tilde{\alpha}_j, \tilde{\rho})$, with $\tilde{d}_j = \pi + (\phi_j + \phi_{j-1})/2$, $\tilde{\alpha}_j = 2\pi + \varepsilon - (\phi_j - \phi_{j-1})$, $\hat{g}_j = \widehat{\mathcal{CH}}_{a_j}(f(\cdot;\phi_j) - f(\cdot;\phi_{j-1}))$. For $\phi_j \le d_\nu - \pi/(2k_\nu)$, $S(\tilde{d}_j, \tilde{\alpha}_j, \tilde{\rho})$ contains a sector with bisecting direction d_ν and opening more than π/k_ν, so $\hat{g}_j \in \mathbb{E}\{z\}_{k_\nu, d_\nu}$, and $(\mathcal{S}_{k_\nu,d_\nu}\,\hat{g}_j)(z) = g_j(z)$. In case $\phi_{j-1} \ge d_\nu + \pi/(2k_\nu)$, a sector of the same opening, but with bisecting direction $d_\nu - 2\pi$, is contained in $S(\tilde{d}_j, \tilde{\alpha}_j, \tilde{\rho})$; hence again $\hat{g}_j \in \mathbb{E}\{z\}_{k_\nu, d_\nu}$, but this time $(\mathcal{S}_{k_\nu,d_\nu}\,\hat{g}_j)(z) = g_j(z\mathrm{e}^{2\pi i})$. Consequently, $\hat{g} = \sum_{j=1}^m \hat{g}_j \in \mathbb{E}\{z\}_{k,d}$. Define

$$f_j(z) = \sum_{\mu=1}^{j} g_\mu(z) + \sum_{\mu=j+1}^{m} g_\mu(ze^{2\pi i}), \quad 0 \le j \le m,\ z \in S_{\phi_j} \cap S_{\phi_{j-1}}.$$

Then $(\mathcal{S}_{k,d}\,\hat{g})(z) = f_j(z)$, for every j with $\phi_j \in I_1$. Moreover, $h(z) = f(z;\phi_j) - f_j(z)$ can be seen to be independent of j, holomorphic and single-valued for $0 < |z| < \tilde{\rho}$, and bounded at the origin. So $\hat{h} = J(h)$ is convergent; hence $\hat{f} = \hat{h} + \hat{g} \in \mathbb{E}\{z\}_{k;d}$, and $S_{k,d}\hat{f} = f(\cdot;\phi_j)$, for every j with $\phi_j \in I_1$. □

With help of the above two propositions, it is now easy to deal with products of multisummable series:

Theorem 52 *Let $\mathbb{E}, \mathbb{F}$ both be Banach spaces, and let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$, $\hat{T} \in \mathcal{L}(\mathbb{E},\mathbb{F})\{z\}_{k,d}$, $\hat{\alpha} \in \mathbb{C}\{z\}_{k,d}$. Then*

$$\begin{aligned} \hat{T}\,\hat{f} &\in \mathbb{F}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{T}\,\hat{f}) = (\mathcal{S}_{k,d}\,\hat{T})\,(\mathcal{S}_{k,d}\,\hat{f}),\\ \hat{\alpha}\,\hat{f} &\in \mathbb{F}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{\alpha}\,\hat{f}) = (\mathcal{S}_{k,d}\,\hat{\alpha})\,(\mathcal{S}_{k,d}\,\hat{f}). \end{aligned}$$

Proof: Without loss in generality, let $k_q > 1/2$. Let $T(z;\phi)$, resp. $\alpha(z;\phi)$, resp. $f(z;\phi)$ be the functions corresponding to $\hat{T}(z)$, resp. $\hat{\alpha}(z)$, resp. $\hat{f}(z)$ by means of Proposition 24. The products $T(z;\phi)f(z;\phi)$ and $\alpha(z;\phi)f(z;\phi)$ then can be checked to have the same properties, and therefore Proposition 25 can be used to complete the proof. □

As for the case of $q = 1$, this shows that $\mathbb{E}\{z\}_{k,d}$ is an algebra, in the case of $\mathbb{E}$ being a Banach algebra, and we can also characterize the invertible elements:

Theorem 53 *Let $\hat{f}, \hat{g}_1, \hat{g}_2 \in \mathbb{E}\{z\}_{k,d}$ be given. If $\mathbb{E}$ is a Banach algebra, then*

$$\hat{g}_1\,\hat{g}_2 \in \mathbb{E}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{g}_1\,\hat{g}_2) = (\mathcal{S}_{k,d}\,\hat{g}_1)\,(\mathcal{S}_{k,d}\,\hat{g}_2).$$

Moreover, if $\mathbb{E}$ has a unit element and $\hat{f}$ has invertible constant term, then

$$\hat{f}^{-1} \in \mathbb{E}\{z\}_{k,d}, \qquad \mathcal{S}_{k,d}\,(\hat{f}^{-1}) = (\mathcal{S}_{k,d}\,\hat{f})^{-1},$$

wherever the right-hand side is defined.

Proof: The first statement follows from Theorem 52, since every element of $\mathbb{E}$ can be regarded as a linear operator on $\mathbb{E}$. For the second one, let $k_q > 1/2$; otherwise, make a change of variable $z = w^p$ with sufficiently large $p \in \mathbb{N}$. Then, let $f(z;\phi)$ be as in Proposition 24. Making ρ and ε smaller if needed, we may arrange that $g(z;\phi) = f(z;\phi)^{-1}$ exists for every $z \in S_\phi$. It then follows by means of Proposition 25 that $g(z;d_1) = (\mathcal{S}_{k,d}\,\hat{g})(z)$ in S_{d_1}, for some $\hat{g}$. Since $g(z;\phi)\,f(z;\phi) \equiv 1$, this shows $\hat{g} = \hat{f}^{-1}$. $\square$

As we have shown above and in Section 10.5, multisummability in a direction satisfies the same rules as k-summability. It is obvious that Theorems 51 and 53 also hold with $\mathbb{E}\{z\}_k$ instead of $\mathbb{E}\{z\}_{k,d}$. However, note that the Main Decomposition Theorem on p. 164 *does not generalize to* $\mathbb{E}\{z\}_k$, as is shown in one of the exercises below. Whether it admits generalization when we consider decompositions into sums of products of series in $\mathbb{E}\{z\}_{k_j}$ seems to be unknown – such decompositions hold, however, for formal solutions of systems (3.1) (p. 37), as was shown in Theorem 43 (p. 134).

Note that the properties of multisummability shown in this chapter, together with Theorem 43 (p. 134), enable us to conclude that every formal fundamental solution of every system (3.1) (p. 37) is multisummable. The same is true even for formal solutions of nonlinear systems, but we shall not prove this here.

Exercises:

1. For $\psi(z) = (1-z)^{-1}\exp[-z^{-1}]$, show that $\hat{f}(z) = \sum_0^\infty f_n\,z^n$, with
$$f_n = \Gamma(1+n/2)\int_0^{1/2} \psi(w) w^{-n-1} dw,$$
is in $\mathbb{E}\{z\}_k$, $k = (2,1)$, and determine all singular multidirections.

2. Let $\hat{f} = \hat{f}_1 + \hat{f}_2$, $\hat{f}_j \in \mathbb{E}\{z\}_{k_j}$, $j = 1,2$, with $k_1 > k_2 > 0$, and let $g = \mathcal{S}\,(\hat{\mathcal{B}}_{\kappa_2} \circ \hat{\mathcal{B}}_{\kappa_1}\,\hat{f})$. Let $d = (d_1, d_2)$ be singular of level 2, and define $h_\pm$ by application of $\mathcal{L}_{\kappa_2}$ to g, with integration along $\arg z = d_2 \pm \varepsilon$, with small $\varepsilon > 0$. Show that $\psi = h_+ - h_-$ is holomorphic in the sector $S = S(\phi, \pi/\kappa_2)$.

3. Show that $\hat{f} \in \mathbb{E}\{z\}_k$, $k = (2,1)$, exists, which is *not* the sum of $\hat{f}_1 \in \mathbb{E}\{z\}_2$ and $\hat{f}_2 \in \mathbb{E}\{z\}_1$. Why does this not contradict Theorem 50?

4. Show that Exercise 5 on p. 104 generalizes to multisummability.

10.8 Optimal Summability Types

It is common to say that a formal power series $\hat{f}$ is *multisummable*, if we can find an admissible k so that $\hat{f} \in \mathbb{E}\{z\}_k$. The following theorem shows that there always exists an optimal choice for k.

Theorem 54 *Let $k = (k_1, \ldots, k_q)$ and $\tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_{\tilde{q}})$ both be admissible, and assume $\hat{f} \in \mathbb{E}\{z\}_k \cap \mathbb{E}\{z\}_{\tilde{k}}$. Then the following holds:*

(a) *If $\{k_1, \ldots, k_q\} \cap \{\tilde{k}_1, \ldots, \tilde{k}_{\tilde{q}}\} = \emptyset$, then $\hat{f}$ converges.*

(b) *In case $\{k_1, \ldots, k_q\} \cap \{\tilde{k}_1, \ldots, \tilde{k}_{\tilde{q}}\} = \{\hat{k}_1, \ldots, \hat{k}_{\hat{q}}\} \neq \emptyset$, assume without loss of generality $\hat{k}_1 > \hat{k}_2 > \ldots > \hat{k}_{\hat{q}} \quad (> 0)$, so that $\hat{k} = (\hat{k}_1, \ldots, \hat{k}_{\hat{q}})$ is a type of multisummability. Then $\hat{f} \in \mathbb{E}\{z\}_{\hat{k}}$.*

Proof: Let $\bar{k} = (\bar{k}_1, \ldots, \bar{k}_{\bar{q}})$ be the type of multisummability corresponding to the union $\{k_1, \ldots, k_q\} \cup \{\tilde{k}_1, \ldots, \tilde{k}_{\tilde{q}}\}$. From Proposition 23 we learn that $\hat{f} \in \mathbb{E}\{z\}_{\bar{k}}$, and there, $\hat{f}$ has no singular directions of level j whenever $\bar{k}_j \notin \{k_1, \ldots, k_q\} \cap \{\tilde{k}_1, \ldots, \tilde{k}_{\tilde{q}}\}$. Hence Proposition 22 resp. Exercise 2 on p. 169 complete the proof. □

Under the assumptions of Theorem 54, let $d = (d_1, \ldots, d_q)$, resp. $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_{\tilde{q}})$, be admissible and nonsingular with respect to k, resp. $\tilde{k}$. In case (a), we then have $(\mathcal{S}_{k,d}\,\hat{f})(z) = (\mathcal{S}_{\tilde{k},\tilde{d}}\,\hat{f})(z) = (\mathcal{S}\,\hat{f})(z)$, for every z where the first two expressions are both defined. In case (b), assume in addition that $k_j = \tilde{k}_\nu$ implies $d_j = \tilde{d}_\nu$, for all j, ν with $1 \le j \le q$, $1 \le \nu \le \tilde{q}$. Define $\hat{d}$ so that $\hat{k}_j = k_\nu \ (= \tilde{k}_{\tilde{\nu}})$ implies $\hat{d}_j = d_\nu \ (= \tilde{d}_{\tilde{\nu}})$. Then Exercise 1 on p. 163 implies $(\mathcal{S}_{k,d}\,\hat{f})(z) = (\mathcal{S}_{\tilde{k},\tilde{d}}\,\hat{f})(z) = (\mathcal{S}_{\hat{k},\hat{d}}\,\hat{f})(z)$, for every z where all three expressions are defined. This shows the following: *While different choices of multidirections in general produce different functions $(\mathcal{S}_{\hat{k},\hat{d}}\,\hat{f})(z)$, it is not possible by choosing a different summability type k to produce even more functions as corresponding sums.*

Exercises:

1. For $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n$ and $\varepsilon = e^{2\pi i/p}$, with integer $p \ge 2$, define $\hat{g}(z) = \hat{f}(z\varepsilon) = \sum_{n=0}^{\infty} f_n \varepsilon^n z^n$. Show that $\hat{f}$ multisummable implies $\hat{g}$ multisummable, and the corresponding optimal types concide.

2. For $\hat{f}$ and p as above, define $\hat{f}_j(z) = \sum_{n=0}^{\infty} f_{np+j} z^n$, $j = 0, \ldots, p-1$. Show that $\hat{f}$ is multisummable if and only if $\hat{f}_j$ is multisummable for every $j = 0, \ldots, p-1$. What about the corresponding optimal types?

3. Given $k_1 > \ldots > k_q > 0$, and defining κ_j accordingly, we say that a subset of $\mathbb{R}^q$ *contains almost all* multidirections $d = (d_1, \ldots d_q)$, provided that d_q can be any value in the half-open interval $[0, 2\pi)$

but finitely many exceptional ones, while given $d_j, \ldots, d_q$, the value d_{j-1} can be any number with $2\kappa_j \, |d_j - d_{j-1}| \le \pi$ but finitely many, for $2 \le j \le q$. Using this terminology, show the following analogue to the Main Decomposition Theorem:

Theorem: *Let $\hat{f}$ be multisummable. Then there exist $k_1 > \ldots > k_q > 0$, so that for almost all multidirections $d = (d_1, \ldots, d_q)$ we have $\hat{f} = \hat{f}_1 + \ldots + \hat{f}_q$, with $\hat{f}_j \in \mathbb{E}\{z\}_{k_j, d_j}$.*

Discuss in which cases we can take $k = (k_1, \ldots, k_q)$ to be the optimal type of $\hat{f}$.

11

Ecalle's Acceleration Operators

Although Ecalle's acceleration operators are in no way special in our presentation of multisummability theory, they are nonetheless important in many applications, owing to their close relation with Laplace transform and convolution of functions. Therefore we now will introduce them and show their main properties.

For real $\alpha > 1$ and complex z, let

$$C_\alpha(z) = \frac{1}{2\pi i} \int_\gamma u^{1/\alpha - 1} \exp[u - z\, u^{1/\alpha}]\, du,$$

with a path of integration γ as in Hankel's integral for the inverse Gamma function: from ∞ along $\arg u = -\pi$ to some $u_0 < 0$, then on the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along this ray. Because of $\alpha > 1$, this integral represents an entire function of z. By termwise integration of $\exp[-z\, u^{1/\alpha}] = \sum_0^\infty (-z)^n u^{n/\alpha}/n!$ and use of Hankel's formula (p. 228), we find the power series expansion

$$C_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\, \Gamma(1 - (n+1)/\alpha)}.$$

Using (B.14) (p. 232) and Theorem 69 (p. 233), one can show that C_α is of exponential order $\beta = \alpha/(\alpha - 1)$ and finite type. For $\alpha = 2$, one can express C_2 in terms of a confluent hypergeometric function, while for $\alpha \to \infty$, $C_\alpha(z) \to \mathrm{e}^{-z}$. By a change of variable $z u^{1/\alpha} = w^{-1}$, and then substituting $z = t^{-1}$, we see that $t^{-1}\, C_\alpha(t^{-1})$ is the Borel transform of index α of $z^{-1} e^{-1/z}$. From Theorem 32 (p. 95) we see that $e(t) = t\, C_\alpha(t)$

therefore is a kernel function of order $\beta = \alpha/(\alpha-1)$, corresponding to the moment function $m(u) = \Gamma(1+u)/\Gamma(1+u/\alpha)$.

11.1 Definition of the Acceleration Operators

For real numbers d and $\tilde{k} > k > 0$, set $\alpha = \tilde{k}/k$. Then the function $e_{\tilde{k},k}(t) = k\, t^k\, C_\alpha(t^k)$ is a kernel function of order $\kappa = k\,\beta = (1/k - 1/\tilde{k})^{-1}$. The corresponding integral operator, denoted by $\mathcal{A}_{\tilde{k},k}$, shall be named the *acceleration operator* with indices $\tilde{k}$ and k, bearing in mind that $\tilde{k} > k$ is always implicitly assumed. One verifies easily that

$$(\mathcal{A}_{\tilde{k},k} f)(z) = z^{-k} \int_0^{\infty(d)} f(t)\, C_\alpha\left((t/z)^k\right) d\, t^k,$$

for $f \in \boldsymbol{A}^{(\kappa)}(S, \mathbb{E})$. Note that our definition of acceleration operators differs slightly from the one used by other authors; this is so to make them match with our definition of Borel and Laplace transforms.

It follows right from the definition that the moment function corresponding to $\mathcal{A}_{\tilde{k},k}$ is $\Gamma(1+u/k)/\Gamma(1+u/\tilde{k})$. This motivates to define the *formal acceleration operator* of a formal power series $\hat{f}(z) = \sum_0^\infty f_n z^n$ by

$$(\hat{\mathcal{A}}_{\tilde{k},k} \hat{f})(z) = \sum_0^\infty z^n f_n \frac{\Gamma(1+n/k)}{\Gamma(1+n/\tilde{k})}. \tag{11.1}$$

The following theorem is a result of our general discussion of integral operators, especially of Theorem 27 (p. 91):

Theorem 55 *For $\tilde{k} > k > 0$ and $1/\kappa = 1/k - 1/\tilde{k}$, let $f \in \boldsymbol{A}^{(\kappa)}(S, \mathbb{E})$ for a sector $S = S(d, \alpha)$, and let $g = \mathcal{A}_{\tilde{k},k} f$, defined in a corresponding sectorial region $G = G(d, \alpha + \pi/k)$. For $s_1 \geq 0$, assume $f(z) \cong_{s_1} \hat{f}(z)$ in S, take $s_2 = 1/\kappa + s_1$, and let $\hat{g} = \hat{\mathcal{A}}_{\tilde{k},k} \hat{f}$. Then*

$$g(z) \cong_{s_2} \hat{g}(z) \quad \textit{in } G.$$

Considering the corresponding moment functions, we obtain from Theorem 31 (p. 94) that $\mathcal{L}_k = \mathcal{L}_{\tilde{k}} * \mathcal{A}_{\tilde{k},k}$. Therefore, Lemma 18 (p. 160) shows that $\mathcal{L}_k = \mathcal{L}_{\tilde{k}} \circ \mathcal{A}_{\tilde{k},k}$ on $\boldsymbol{A}^{(k)}(S, \mathbb{E})$:

Theorem 56 *In addition to the assumptions of Theorem 55 (p. 176), let f be of exponential growth not more than k. Then $g = \mathcal{A}_{\tilde{k},k} f$ is holomorphic and of exponential growth not more than $\tilde{k}$ in $\tilde{S} = S(d, \alpha + \pi/\kappa)$, and $\mathcal{L}_{\tilde{k}} g = \mathcal{L}_k f$.*

The inverse of the acceleration operators are briefly discussed in the following exercises.

Exercises: In the following exercises, fix $\tilde{k} > k > 0$, and set $\alpha = \tilde{k}/k$, $1/\kappa = 1/k - 1/\tilde{k}$. Moreover, let $D_\alpha(z) = \sum_{n=0}^{\infty} z^n\,\Gamma(1+n/\alpha)/n!$, for $z \in \mathbb{C}$.

1. Show that D_α, defined above, is an entire function of exponential order $\beta = \alpha/(\alpha-1)$ and finite type. Moreover, prove the integral representation $D_\alpha(z) = \int_0^{\infty(d)} \exp[zx^{1/\alpha} - x]\,dx$, for every z, and $d \in (-\pi/2, \pi/2)$.

2. Prove[1]
$$D_\alpha(-1/z) \cong_1 \alpha \sum_{n=0}^{\infty} (-1)^n z^{\alpha(n+1)} \frac{\Gamma(\alpha(n+1))}{n!}$$
in $S(0, \pi(1+1/\alpha))$.

3. For a sectorial region $G = G(d,\phi)$ of opening more than π/κ, and $f \in \boldsymbol{H}(G,\mathbb{E})$ continuous at the origin, define the *deceleration operator* $\mathcal{D}_{\tilde{k},k}$ with indices $\tilde{k}$ and k by
$$(\mathcal{D}_{\tilde{k},k} f)(u) = \frac{1}{2\pi i} \int_{\gamma_\kappa(\tau)} z^k f(z) D_\alpha\big((u/z)^k\big)\, d\,z^{-k},$$
with $\gamma_\kappa(\tau)$ as in Section 5.2. Show that $\mathcal{D}_{\tilde{k},k}$ and $\mathcal{A}_{\tilde{k},k}$ are inverse to one another in the sense of Section 5.7.

11.2 Ecalle's Definition of Multisummability

Let any summability type $k = (k_1, \ldots, k_q)$ be given. By definition, the corresponding κ_j satisfy $\kappa_1 = k_1$, $1/\kappa_j = 1/k_j - 1/k_{j-1}$, $2 \le j \le q$. Since for k-summability we may choose any integral operators T_j having the required orders κ_j, we *may take* $T_1 = \mathcal{L}_{k_1}$, $T_j = \mathcal{A}_{k_{j-1},k_j}$, $2 \le j \le q$. These operators were used by *Ecalle* in his original definition of multisummability. It is a very natural choice for the following reason: *By looking at the corresponding moment functions, we find*
$$\mathcal{L}_{k_1} * \mathcal{A}_{k_1,k_2} * \ldots * \mathcal{A}_{k_{q-1},k_q} = \mathcal{L}_{k_q}.$$
Therefore, we may say that the iterated operator $\mathcal{L}_{k_1} \circ \mathcal{A}_{k_1,k_2} \circ \ldots \circ \mathcal{A}_{k_{q-1},k_q}$ *is a natural extension of the operator* $\mathcal{L}_{k_q}$ *to a wider class of functions. Hence,*

[1] Here we use a more general definition of Gevrey asymptotics: For $\alpha, k > 0$, we say that $f(z) \cong_s \sum_{n=0}^{\infty} f_n\, z^{\alpha n}$ in $S(d,\beta,r)$ if and only if $f(z^{1/\alpha}) \cong_{s\alpha} \sum_{n=0}^{\infty} f_n\, z^n$ in $S(\alpha d, \alpha\beta, r^\alpha)$. Compare this to Exercise 2 on p. 72 to see that this is correct in case of α being a natural number.

k-summability differs from k_q-summability in using this extended Laplace operator instead of the usual one. Another good reason for prefering this choice of operators over others lies in their strong relation with convolution of functions. This will be discussed next.

Exercises: Let a fixed summability type $k = (k_1, \dots, k_q)$ be given, and let the corresponding operators be chosen as above.

1. Show that for $\hat{f} \in \mathbb{E}[[z]]$, the series $\hat{f}_j$ defined in Exercise 2 on p. 162 are equal to $\sum f_n z^n / \Gamma(1 + n/k_j)$.

2. Let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$, for a multidirection $d = (d_1, \dots, d_q)$ with $d_1 = \dots = d_q$. For $s \geq 1/k_q$, define $f(z; s)$ as the sum of the convergent series $\hat{\mathcal{B}}_{1/s} \hat{f}$. Interpret k-summability of $\hat{f}$ in this multidirection as a continuation of $f(z; s)$ with respect to s to the interval $s \geq 1/k_1$.

11.3 Convolutions

The main advantage of acceleration operators over general ones of the same order lies in the fact that they behave well with respect to convolution of functions, which we shall introduce and discuss now. *To do so, we shall for simplicity of notation assume that $\mathbb{E}$ is a Banach algebra*, although similar results hold in all cases where the products of the functions resp. power series occuring are defined, e.g., in the situation of Theorem 14 (p. 67).

For arbitrary $k > 0$, let $s = 1/k$. Given any $\hat{f}, \hat{g} \in \mathbb{E}[[z]]$, consider $\hat{h} = \hat{\mathcal{B}}_k [(\hat{\mathcal{L}}_k \hat{f})(\hat{\mathcal{L}}_k \hat{g})]$. The coefficients h_n of this series are given by

$$h_n = \frac{1}{\Gamma(1 + sn)} \sum_{m=0}^{n} f_{n-m} \, \Gamma(1 + s(n-m)) \, g_m \, \Gamma(1 + sm), \quad n \geq 0.$$

Using the *Beta Integral* (p. 229) and integrating termwise, this identity can be written as $\hat{h}(z^s) = (d/dz) [\int_0^z \hat{f}((z-t)^s) \, \hat{g}(t^s) dt]$. The series $\hat{h}$ will be called *formal convolution of index k of $\hat{f}$ and $\hat{g}$*, and we write $\hat{h} = \hat{f} *_k \hat{g}$. Now, let G be a sectorial region and consider two functions $f, g \in \boldsymbol{H}(G, \mathbb{E})$ which are continuous at the origin. We define a function $h(z)$, holomorphic in G, by

$$h(z^s) = \frac{d}{dz} \Big[\int_0^z f((z-t)^s) \, g(t^s) dt \Big], \quad z^s \in G, \tag{11.2}$$

and call h *the convolution* of f and g (with index k). In shorthand we shall write $f *_k g$ for this function h.

The convolution operator is well behaved with respect to Gevrey asymptotics, as is shown in the following lemma:

Lemma 22 *For k, f, g, and G as above, assume $f(z) \cong_{s_1} \hat{f}(z)$, $g(z) \cong_{s_1} \hat{g}(z)$ in G, for some $s_1 > 0$. Then we have $(f *_k g)(z) \cong_{s_1} (\hat{f} *_k \hat{g})(z)$ in G. In case G is a sector of infinite radius and f, g are of exponential growth not more than κ, with some $\kappa > 0$, then so is $f *_k g$.*

Proof: By assumption, for every closed subsector $\bar{S}$ of G there exist $c, K > 0$ so that $|r_f(z,N)|, |r_g(z,N)| \le c\,K^N \Gamma(1 + s_1 N)$ for every $N \ge 0$ and $z \in \bar{S}$. Using the Beta Integral, one can show for $z^s \in \bar{S}$ and $h = f *_k g$, $\hat{h} = \hat{f} \hat{*}_k \hat{g}$:

$$\begin{aligned} z^{sN} r_h(z^s, N) &= \frac{d}{dz} \int_0^z \Bigg[f\left((z-t)^s\right) t^{sN} r_g(t^s, N) \\ &+ \sum_{m=0}^{N-1} g_m \; (z-t)^{sm} t^{s(N-m)} \; r_f(t^s, N-m) \Bigg] \, dt. \end{aligned}$$

Using the above estimates, we find

$$\left| \int_0^z f\left((z-t)^s\right) t^{sN} \; r_g(t^s, N) dt \right| \le c^2 K^N \Gamma(1 + N\,s_1) |z|^{1+sN} / (1 + s_1\,N),$$

whereas, using Exercise 1 on p. 41,

$$\begin{aligned} &\left| \int_0^z \sum_{m=0}^{N-1} g_m t^{sm} (z-t)^{s(N-m)} \; r_f\left((z-t)^s, N-m\right) dt \right| \\ &\qquad\qquad \le \frac{c^2\,K^N |z|^{1+sN}}{1+sN} \sum_{m=0}^{N-1} \Gamma(1 + m s_1)\, \Gamma(1 + (N-m)s_1). \end{aligned}$$

This shows altogether for sufficiently large $\hat{c}, \hat{K} > 0$, independent of N, z as above, and $N \ge 0$

$$\left| \int_0^z w^{sN} \; r_h(w^s, N) dw \right| \le \hat{c} \hat{K}^N |z|^{1+sN} \Gamma(1 + N s_1).$$

Using Cauchy's formula for the first derivative, one can see that this implies $h(z) \cong_{s_1} \hat{h}(z)$ in G. An elementary estimate can be used to show that h is of the desired exponential growth, in case f and g are. □

As we indicated above, the acceleration operators are well behaved with respect to convolutions. To prove this, we first show that the Laplace operator of order k maps convolutions of the same index onto products:

Theorem 57 *Assume that $\mathbb{E}$ is a Banach algebra. Let S be a sector of infinite radius, and let $k > 0$ be arbitrarily given. Moreover, let f, g be $\mathbb{E}$-valued functions, holomorphic in S, continuous at the origin, and of exponential growth not more than k. Then*

$$\mathcal{L}_k(f *_k g) = (\mathcal{L}_k f)(\mathcal{L}_k g).$$

Proof: Setting $h = f *_k g$, $\tilde{h} = \mathcal{L}_k h$, $s = 1/k$, and choosing d appropriately, we have $\tilde{h}(z^s) = \int_0^{\infty(d)} \exp[-w/z]\, h(w^s)\, dw$. Inserting for $h(w^s)$, integrating by parts, and then interchanging the order of integration shows $\tilde{h} = (\mathcal{L}_k f)(\mathcal{L}_k g)$. □

We now deal with the acceleration operators:

Theorem 58 *Assume that $\mathbb{E}$ is a Banach algebra. Let S be a sector of infinite radius, let $\tilde{k} > k > 0$ be arbitrarily given, and take $1/\kappa = 1/k - 1/\tilde{k}$. Moreover, let f, g be $\mathbb{E}$-valued functions, holomorphic in S, continuous at the origin, and of exponential growth not more than κ. Then*

$$\mathcal{A}_{\tilde{k},k}(f *_k g) = (\mathcal{A}_{\tilde{k},k} f) \ *_{\tilde{k}} \ (\mathcal{A}_{\tilde{k},k} g).$$

Proof: With $\tilde{h} = (\mathcal{A}_{\tilde{k},k} f) *_{\tilde{k}} (\mathcal{A}_{\tilde{k},k} g)$, $\alpha = \tilde{k}/k$, $\tilde{s} = 1/\tilde{k}$, and suitable d, we find

$$\tilde{h}(z^{\tilde{s}}) = \int_0^{\infty(d)} f(u) \int_0^{\infty(d)} g(w) k(z^{1/\alpha}, w, u)\, dw^k\, du^k,$$

$$k(z^{1/\alpha}, w, u) = \frac{d}{dz} \int_0^z (z-t)^{-1/\alpha}\, C_\alpha\Big(\frac{u^k}{(z-t)^{1/\alpha}}\Big)\, t^{-1/\alpha}\, C_\alpha\Big(\frac{w^k}{t^{1/\alpha}}\Big) dt.$$

For fixed w and u, $k(z, w, u)$ is the convolution of index α of the functions $z^{-1}\, C_\alpha(u^k/z)$ and $z^{-1}\, C_\alpha(w^k/z)$, which in turn are the Borel transform of order α of $z^{-1} e^{-u^k/z}$ and $z^{-1} e^{-w^k/z}$. From the previous theorem we therefore find that $k(z, w, u)$ is the Borel transform of the same order of the product $z^{-2} e^{-(u^k+w^k)/z} = -(\partial/\partial u^k) z^{-1} e^{-(u^k+w^k)/z}$. This in turn shows $k(z, w, u) = -(\partial/\partial u^k) z^{-1}\, C_\alpha((u^k + w^k)/z)$. So we obtain, replacing z by z^α, hence $z^{\tilde{s}}$ by z^s with $s = 1/k$, and making corresponding changes of variables in the integrals:

$$\tilde{h}(z^s) = \frac{-1}{z} \int_0^{\infty(kd)} \left[\int_0^{\infty(kd)} f(u^s) \frac{\partial}{\partial u} C_\alpha((u+w)/z)\, du \right] g(w^s)\, dw.$$

Setting $u = t - w$, hence $(\partial/\partial u) = (\partial/\partial t)$, and interchanging the order of integration, followed by integration by parts then gives

$$\tilde{h}(z^s) = \frac{1}{z} \int_0^{\infty(kd)} C_\alpha(t/z) \frac{d}{dt} \int_0^t f((t-w)^s) g(w^s)\, dw\, dt.$$

This, however, is equivalent to $\tilde{h} = \mathcal{A}_{\tilde{k},k}(f *_k g)$. □

Exercises: As above, assume that $\mathbb{E}$ is a Banach algebra. Let S be a sectorial region, and let f, g be $\mathbb{E}$-valued, holomorphic in G, and continuous at the origin.

1. Show that $\lim_{k\to\infty}(f *_k g)(z) = f(z)\, g(z)$, $z \in G$.

2. Let $\mathbb{E}$ have unit element e, and set $f_{\alpha,k}(z) = z^\alpha e/\Gamma(1+\alpha/k)$, $\alpha \in \mathbb{R}_0$. Show that $f_{0,k} *_k g = g$, while

$$(f_{\alpha,k} *_k g)(z) = \int_0^z \frac{(z^k - t^k)^{\alpha/k-1}}{\Gamma(\alpha/k)}\, g(t)\, dt^k, \quad \alpha > 0.$$

11.4 Convolution Equations

We briefly mention the following application of the previous results: Let $\mathbb{E}$ be a Banach algebra with unit element e, let $a_n \in \mathbb{E}$ be so that $a(z) = e - \sum_1^\infty a_n\, z^n$ converges for $|z| < \rho$, with $\rho > 0$. Moreover, let $\hat{f} \in \mathbb{E}\{z\}_{k,d}$, for some admissible $k = (k_1, \ldots, k_q)$ and $d = (d_1, \ldots, d_q)$. For a given natural number r, the inhomogeneous differential equation

$$z^{r+1}\, x' + a(z)\, x = \hat{f}(z) \tag{11.3}$$

then has a unique formal solution $\hat{x}(z) = \sum_0^\infty x_n\, z^n$. Its coefficients are given by the recursion $x_n = f_n - (n-r)\, x_{n-r} + \sum_{m=0}^{n-1} a_{n-m}\, x_m$, $n \geq 0$, with $x_{-r} = \ldots = x_{-1} = 0$. We wish to show *that this solution again is multisummable.* For simplicity of notation, let us assume that for some, necessarily unique, j we have $k_j = r$; if this were not so, we had to introduce a summability type $\tilde{k}$ corresponding to the set $\{\, r, k_1, \ldots, k_q\,\}$. To deal with the above question, it is convenient to make the *canonical choice of operators* $\mathcal{L}_{k_1}, \mathcal{A}_{k_1,k_2}, \ldots, \mathcal{A}_{k_{q-1},k_q}$, so that $\hat{f}_j = \hat{\mathcal{B}}_{k_j} \hat{f}$. For notational convenience, we here set $k_0 = \infty$ and $\mathcal{A}_{k_0,k_1} = \mathcal{L}_{k_1}$.

Corresponding to $\hat{f}$, there exist functions $f_0, \ldots, f_q$ and formal power series $\hat{f}_0, \ldots, \hat{f}_q$ as in Exercise 2 on p. 162. Each f_j is holomorphic in a sectorial region $G(d_{j+1}, \alpha_j)$, with $\alpha_j > \pi/\kappa_{j+1}$ (in case $j \leq q-1$), resp. in a disc about the origin (in case $j = q$). Moreover, for $j \geq 1$, the function $f_j(z)$ admits holomorphic continuation into a sector $S(d_j, \varepsilon)$ and is of exponential growth at most κ_j there. Any two consecutive functions f_j, f_{j+1} are related via the operator $\mathcal{A}_{k_j,k_{j+1}}$. Our goal is to prove the existence of functions x_j, corresponding to $\hat{x}$, and having the same features. To do so, we investigate the identities

$$(\delta - r)\, (f_{r,k_j} *_{k_j} x_j)(z) + (a_j *_{k_j} x_j)(z) = f_j(z), \quad 1 \leq j \leq q, \tag{11.4}$$

where a_j denotes the Borel transform of order k_j of a, δ stands for the operator $z\,(d/dz)$, and f_{r,k_j} is as in Exercise 2 on p. 181. These equations *formally hold* when we replace a_j by its power series, f_j by $\hat{f}_j$, and x_j by $\hat{\mathcal{B}}_{k_j}\hat{x}$. Without going into detail, we state the following:

For j with $k_j \le r$, this identity is nothing but a Volterra-type integral equation for the function x_j. The usual iteration technique shows that every such equation has a unique solution that has the same features as f_j: It is holomorphic in the region $G(d_{j+1},\alpha_j)$ (in case $j \le q-1$), *resp. in a disc about the origin* (in case $j = q$). *Moreover, for $j \ge 1$, the function x_j admits holomorphic continuation into the sector $S(d_j,\varepsilon)$ and is of exponential growth at most κ_j there. It then follows from uniqueness of the solutions that, for every such j, we have $x_j = \mathcal{A}_{k_j,k_{j+1}}\, x_{j+1}$, in case $j < q$.*

For j with $k_j > r$, identity (11.4) *is of a different nature: It can be interpreted as a singular integral equation for x_j, hence the standard iteration method fails. However, once a solution x_j is known, say, for z in a sector of finite radius, then* (11.4), *for $j \ge 1$, can still serve to show continuation of x_j into $S(d_j,\varepsilon)$, and get a growth estimate allowing application of $\mathcal{A}_{k_{j-1},k_j}$. This operator then defines the next function x_{j-1}, and in this fashion all $x_0,\ldots,x_q$ are obtained.*

While in the general theory of multisummability the choice of operators was of no real importance, here it is essential to choose the acceleration operators: For others, there is no notion comparing to the convolution of functions, making it impossible to study the equations that correspond to (11.3) by formal application of the inverse operators T_j^{-1}.

Exercises: Use the notation introduced above.

1. For $\tilde{k} > k > 0$, show $\delta(\mathcal{L}_k f) = \mathcal{L}_k(\delta f)$, $\delta(\mathcal{A}_{\tilde{k},k} f) = \mathcal{A}_{\tilde{k},k}(\delta f)$, whenever f is holomorphic and of appropriate growth in a sector of infinite radius.

2. Formally, show $(a_j *_{k_j} \hat{x}_j)(z) = \hat{x}_j(z) - \int_0^z b_j(z,t)\,\hat{x}_j(t)\,dt$, with
$$b_j(z,t) = \sum_1^\infty a_n\,(z^{k_j} - t^{k_j})^{n/k_j-1}/\Gamma(n/k_j).$$

3. Show $(\delta - r)\,(f_{r,r} *_r \hat{x})(u) = r\,[u^r\,\hat{x}(u) - \int_0^u \hat{x}(t)\,dt^r]$, and write (11.4), for $k_j = r$, as a Volterra integral equation.

12
Other Related Questions

In this chapter, we mention some additional results related to the theory of either multisummability or ODE in the complex plane:

In the first section we shall give necessary and sufficient conditions for so-called *matrix summability methods* to be stronger than multisummability. These conditions are very much analogous to the classical characterization of regular summability methods. They may also be viewed as analogues of conditions characterizing what is called *power series regularity*, meaning that a summation method sums convergent power series, *inside* their disc of convergence, to the *correct sum*. So far, only one matrix method is known to be stronger than multisummability. On the other hand, for the subclass of *power series methods* it has been shown in [32] that none of them can have this property.

In the second section, we show in which sense a system of ODE of Poincaré rank $r \geq 2$ is equivalent to one of rank 1, but higher dimension. This sometimes can be useful in generalizing results for systems of rank $r = 1$ to the general case.

Sections 12.3 and 12.4 deal with two different but, as far as the methods used are concerned, intimately related problems that both can roughly be characterized as follows: Given a certain class of systems of ODE, together with a number of parameters that, at least theoretically, can be computed from every such system, then are these parameters *free* in the sense that they can independently take on every value? For the *Riemann-Hilbert problem*, the class of systems under consideration are the Fuchsian systems, and the parameters are their monodromy data. For the problem of *Birkhoff reduction* the systems are those with polynomial coefficient matrix, and the parameters are the so-called *invariants* associated to the singular point at infinity which is assumed to have Poincaré rank $r \geq 1$. Both problems have been believed to be entirely solved for quite some time, but we shall explain that the answer to the first one is, in fact negative, while the second one is still partially open.

The final section then deals with the *central connection problem*: Think of a function given by a convergent power series; then what can be said about its behavior on the boundary of the disc of convergence? If the function satisfies a system of ODE, then its behavior is essentially determined – except for some unknown constants, called the *central connection coefficients*. The computation of these coefficients is what this problem is all about, and we shall show in which sense this can be done.

12.1 Matrix Methods and Multisummability

We recall from Chapter 6 the following notion: Given an infinite matrix $A = (a_{mn})$, with $a_{mn} \in \mathbb{C}$ for $m, n \in \mathbb{N}_0$, we call a formal power series $\hat{f}(z) = \sum f_n\, z^n \in \mathbb{E}\,[[z]]$ *A-summable in a direction $d \in \mathbb{R}$*, if there exists a sectorial region $G = G(d, \alpha)$ of opening $\alpha > 0$, such that the following two conditions hold:

1) The series $f_m(z) = \sum_{n=0}^{\infty} a_{mn} f_n\, z^n$ converge in discs that all contain G, for every $m \in \mathbb{N}_0$.

2) The limit $f(z) = \lim_{m\to\infty} f_m(z)$ exists uniformly on every closed subsector of G.

The so defined function f, which is holomorphic on G, will be called *the A-sum* of $\hat{f}$ on G. The set of all formal power series that are A-summable in direction d will be denoted by $\mathbb{E}\,\{z\}_{A,d}$, and we write $\mathcal{S}_{A,d}\ \hat{f}$ for the A-sum of $\hat{f}$.

In this context, it is natural to call a matrix A *weakly p-regular* (with p being short for power series), if every convergent series is A-summable in every direction d to the *correct* sum. A necessary and sufficient condition for this to hold is given in the exercises. Observe, however, that weak p-regularity is not the same as power series regularity: Let $\hat{f}(z)$ converge, say, for $|z| < r$. Then for a weakly p-regular matrix we have that to every d there exists $G(d, \alpha)$ so that the f_m converge to $f = \mathcal{S}\,\hat{f}$ on $G(d, \alpha)$, and the union of these regions may be a proper subset of the disc of convergence; see the exercises below that this can occur. The definition of power series regular matrices, however, requires this union to be the full disc.

The following problem has been solved by *Beck* [29, 50] for the case of $\mathbb{E} = \mathbb{C}$, but the proofs carry over to a general Banach space: Characterize those matrices A for which the following *comparison condition* (C) holds:

(C) For every Banach space $\mathbb{E}$, every $k > 1/2$ and every $d \in \mathbb{R}$, we have $\mathbb{E}\{z\}_{k,d} \subset \mathbb{E}\{z\}_{A,d}$, and $(\mathcal{S}_{k,d}\,\hat{f})(z) = (\mathcal{S}_{A,d}\,\hat{f})(z)$ for every $\hat{f} \in \mathbb{E}\{z\}_{k,d}$ and all z where both sides are defined.

Because of $\mathbb{E}\{z\} \subset \mathbb{E}\{z\}_{k,d}$, we see that weak p-regularity is a necessary condition for (C), while power series regularity is not. For that reason, *we shall from now on only consider matrices A that are weakly p-regular.* Also, note that $(\mathcal{S}_{k,d}\,\hat{f})(z)$ always is holomorphic on a region of opening larger than π/k and bisecting direction d, while $(\mathcal{S}_{A,d}\,\hat{f})(z)$ is, in general, only defined close to the bisecting ray.

To give necessary and sufficient conditions for (C), we introduce the following terminology: For $m \ge 0$, we define $k_m(z) = \sum_{n=0}^{\infty} a_{mn}\, z^n$. Then we say that $A = (a_{mn})$ satisfies the *regularity condition* (R), if the following holds:

(R) The functions $k_m(z)$, $m \in \mathbb{N}_0$, are all entire, and converge compactly to $(1-z)^{-1}$, for $m \to \infty$ and every z in the sector $S(\pi, 2\pi)$.

Moreover, we say that $A = (a_{mn})$ satisfies the *order condition* (O), if the $k_m(z)$ are all entire functions of exponential order $\le 1/2$. Finally, we say that $A = (a_{mn})$ satisfies the *growth condition* (G), if the following holds:

(G) The functions are all entire, and for every $k > 1/2$ and every σ with $1/k < \sigma < 2$ there exist $c, K > 0$ such that $|k_m(z)| \le c\, e^{K\,|z|^k}$, for every $z \in S(\pi, (2-\sigma)\pi)$ and every $m \in \mathbb{N}_0$.

Observe that the growth condition becomes meaningless for $k \le 1/2$, since then the interval for σ is empty. This is why we here restrict ourselves to $k > 1/2$. Also note that the constants c, K in the estimate are independent of m.

Let a weakly p-regular matrix A be given and assume that (C) holds. For every $k > 1/2$ and d with $0 < d < 2\pi$, the formal series $\hat{f}_k(z) = \sum_0^{\infty} \Gamma(1+n/k)\, z^n$ is in $\mathbb{C}\{z\}_{k,d}$, and hence must be A-summable in every such direction d. This implies that the power series $\sum_0^{\infty} a_{nm}\Gamma(1+n/k)\, z^n$ must have a positive radius of convergence, for every such k and every $m \ge 0$. From this we conclude the existence of $c, K > 0$, depending on m and k but independent of n, so that $|a_{mn}| \le c\, K^n/\Gamma(1+n/k)$ for every $n \ge 0$. Thus, the order condition (O) follows. Moreover, by definition of A-summability we have the existence of $\varepsilon, r > 0$, depending on d and k, so that the functions $f_{m,k}(z) = \sum_0^{\infty} a_{nm}\Gamma(1+n/k)\, z^n$, for $m \to \infty$, converge uniformly on $\bar{S}(d, \varepsilon)$, for every d as above. A compactness argument then shows uniform convergence, hence boundedness, of the $f_{m,k}$ on arbitrary closed subsectors $\bar{S}$ of $S(\pi, 2\pi)$. Since $k_m = \mathcal{B}_k f_{m,k}$, we can use an estimate as in the proof of Theorem 24 (p. 82) to show (G). Finally, interchanging Borel transform and limit, we can conclude that the kernel functions $k_m(z)$

converge to $(1-z)^{-1}$, and convergence is locally uniform, in $S(\pi, (2-1/k)\pi)$. Hence, using that k can be taken arbitrarily large we see that (R) holds. So in shorthand notation, we have shown that (C) implies (O), (R), and (G). The converse also holds, as we now show:

Theorem 59 *Let a weakly p-regular infinite matrix A be given. Then (C) holds if and only if (R), (O), and (G) are satisfied.*

Proof: One direction of the proof has already been given, so we now assume that (R), (O), and (G) are satisfied. For $d \in \mathbb{R}$ and $k > 1/2$, consider a series $\hat{f} \in \mathbb{E}\{z\}_{k,d}$. As shown in the proof of Theorem 41 (p. 120), we can decompose $\hat{f}$ into a convergent series plus finitely many others which are moment series; so without loss of generality we can restrict $\hat{f}$ to have coefficients of the form (7.1) (p. 116), with $\psi \in \boldsymbol{A}_{1/k,0}(G, \mathbb{E})$, and $d + \pi/(2k) < \arg a < d + (2 - 1/(2k))\pi$; compare Remark 9 (p. 117) to see that then $\hat{f} \in \mathbb{E}\{z\}_{k,d}$. In this case, we have

$$f_m(z) = \sum_{n=0}^{\infty} a_{mn}\, f_n\, z^n = \frac{1}{2\pi i} \int_0^a \psi(u)\, k_m(z/u) \frac{du}{u},$$

the interchange of summation and integration being justified because of (O). For $\arg u$ sufficiently close to d, we can then use (G) to justify interchanging integration and limit as $m \to \infty$ to obtain with help of (R):

$$\lim_{m\to\infty} f_m(z) = \frac{1}{2\pi i} \int_0^a \frac{\psi(u)}{u - z} du,$$

and the right-hand side is equal to $\mathcal{S}_{k,d}\,\hat{f}$. □

Let now any multisummability type $k = (k_1, \ldots, k_q)$ be given, assuming $k_1 > \ldots > k_q > 1/2$. Then the parameters κ_j, given by $\kappa_1 = k_1$, $1/\kappa_j = 1/k_j - 1/k_{j-1}$, $2 \le j \le q$, automatically are larger than $1/2$, so that the Main Decomposition Theorem (p. 164) applies. This shows that the above theorem immediately generalizes to multisummable series – *however, we have to restrict to multidirections $d = (d_1, \ldots, d_q)$ with $d_1 = \ldots = d_q$,* because otherwise there may be no common sector on which the A-sum of a multisummable series can be defined.

Not very many matrix methods seem to satisfy the conditions (O), (R), (G): *Jurkat* [144] studied the matrices $J_\alpha = (j_{mn}(\alpha))$ with $j_{mn}(\alpha) = \exp[-\delta_m\, \lambda(\alpha n)]$, where δ_m may be any positive sequence tending to 0 as $m \to \infty$, α is a positive real parameter, and

$$\lambda(u) = u \log(u + 3) \log\log(u + 3).$$

This method had already been introduced by *Hardy* [113] for summation of special power series with rapidly growing coefficients. It is a variant of what is called *Lindelöf's* methods, useful for computing holomorphic continuation. Jurkat showed that his method satisfies all three conditions, so that (C) follows. So far, this is the only method known to have this property. On the other hand, *Braun* in [32] showed that *no power series method can have property* (C).

Exercises: In the following exercises, consider a fixed matrix $A = (a_{mn})$ that may not be weakly p-regular, and define $k_m(z)$ as above, assuming convergence for $|z| < r$, with $r > 0$ independent of m.

1. For $f_m(z) = \sum_0^\infty a_{mn} f_n\, z^n$, assume convergence for $|z| < \rho$, with $0 < \rho$, independent of m. Derive the integral representation $f_m(z) = (1/2\pi i) \oint_{|u|=\tilde\rho} f(u)\, k_m(z/u)\, (du/u)$, for $\tilde\rho < \rho$ and $|z| < \tilde\rho\, r$.

2. Show that A is weakly p-regular if and only if some r with $0 < r \le 1$ exists, for which $k_m(z)$ converge locally uniformly to $(1-z)^{-1}$ on $D(0,r)$. Conclude that power series regularity is equivalent to the same with $r = 1$.

3. For $a \in \mathbb{C}$ with $|a| = 1$, let $a_{mn} = \mathrm{e}^{-am} \sum_{j=n}^\infty (am)^j/j!$, $m, n \ge 0$. Use the previous exercise to conclude that this A is weakly p-regular if and only if a has positive real part, and power series regular if and only if $a = 1$.

12.2 The Method of Reduction of Rank

In this section we show that in some sense a system (3.1) (p. 37) of Poincaré rank $r \ge 2$ is equivalent to one of rank $r = 1$, which will be called the *rank-reduced* system. The process of rank-reduction has been used by *Poincaré* [222] and *Birkhoff* [53] in representing certain solutions of systems of higher rank as Laplace integrals. Also see *Turrittin* [270], *Lutz* [174], *Balser, Jurkat, and Lutz* [38], and *Schäfke and Volkmer* [242].

Let a system (3.1), with $r \ge 2$, be given, let $\varepsilon_r = \exp[2\pi i/r]$. With 0_ν, resp. I_ν denoting the zero, resp. identity, matrix of dimension ν, define the $r\nu \times r\nu$ matrices $D = r^{-1}\,\mathrm{diag}\,[0_\nu, I_\nu, 2\,I_\nu, \ldots, (r-1)\,I_\nu]$, and

$$U = \begin{bmatrix} 0_\nu & 0_\nu & \ldots & 0_\nu & I_\nu \\ I_\nu & 0_\nu & \ldots & 0_\nu & 0_\nu \\ \vdots & \vdots & & \vdots & \vdots \\ 0_\nu & 0_\nu & \ldots & I_\nu & 0_\nu \end{bmatrix}.$$

For an arbitrary fundamental solution $X(z)$ of (3.1), set

$$Y(z) = z^{-D} U^{-1} \mathrm{diag}\,[X(z^{1/r}), X(\varepsilon_r\, z^{1/r}), \ldots, X(\varepsilon_r^{r-1}\, z^{1/r})] U\, z^D.$$

With some patience, one can then verify that $Y(z)$ is a fundamental solution of a system $z\,y' = B(z)\,y$, $B(z) = \sum_{n=0}^{\infty} B_n\, z^{-n}$, with coefficients of the following form:

$$B_n = \begin{bmatrix} A_{rn} & A_{rn+1} & \dots & A_{rn+r-2} & A_{rn+r-1} \\ A_{rn-1} & A_{rn} & \dots & A_{rn+r-3} & A_{rn+r-2} \\ \vdots & \vdots & & \vdots & \vdots \\ A_{rn-r+1} & A_{rn-r+2} & \dots & A_{rn-1} & A_{rn} \end{bmatrix}, \quad n \geq 0,$$

where $A_{-r+1} = \ldots = A_{-1} = 0$, and the others are as in (3.1). In particular, the new system has Poincaré rank one. It is named the *rank-reduced system corresponding to* (3.1). Note that the expansion of $B(z)$ converges for $|z| > \rho^{1/r}$, with $\rho > 0$ as in (3.1). For many more formulas relating (3.1) and its rank-reduced system, see [38].

12.3 The Riemann-Hilbert Problem

Let n distinct complex numbers a_j be given, which for notational convenience are assumed to be nonzero. Consider linear systems of ODE of the form $x' = A(z)\,x$, for $z \in G = \mathbb{C} \setminus \{a_1, \ldots, a_n\}$, and $A(z) = \sum_{j=1}^{n}(z - a_j)^{-1}A_j$, for given matrices $A_j \in \mathbb{C}^{\,\nu\times\nu}$. Obviously, this is a system with singular points at $a_1, \ldots, a_n$ and infinity, which all are singularities of first kind; compare the discussion at the end of Section 1.6 on how to determine the type of singularity at infinity. Every system satisfying the above requirements will be named a *Fuchsian system.*

Let such a Fuchsian system be given. Consider paths γ_j in G, originating from the origin and going to points "near" a_j in one way or another, then encircling a_j in the positive sense along a circle of small radius, and retracing themselves back to the origin. According to results from Chapter 1, there is a unique fundamental solution $X(z)$ of our Fuchsian system satisfying $X(0) = I$, holomorphic in some disc about the origin. We perform its holomorphic continuation along the path γ_j, ending with a fundamental solution $X_{\gamma_j}(z)$, which will in general be different from $X(z)$. In any case, there exists an invertible constant matrix C_{γ_j} so that $X_{\gamma_j}(z) = X(z)\,C_{\gamma_j}$. We refer to C_{γ_j} as *the jth monodromy factor*. Note that we do not consider a monodromy factor corresponding to a path encircling infinity, since such a matrix can be expressed as a product, in a suitable order, of the other ones. These matrices generate a group, called *the monodromy group* of the system, but this is of no importance here.

Obviously, a Fuchsian system is given in terms of $n\,\nu^2$ free parameters, namely, the matrices A_j. The monodromy matrices carry the same number of parameters, and therefore it makes sense to ask the following question:

- Given the points a_j, the paths γ_j, and invertible matrices C_{γ_j}, are there matrices A_j, so that the corresponding Fuchsian system has the C_{γ_j} as its monodromy factors?

The above problem is usually referred to as the *Riemann-Hilbert problem*, or *Hilbert's 21st problem*. It was believed to have been positively solved by *Plemelj* [221], but *Bolibruch* [58, 59, 63] in 1989 showed that the answer is in fact negative by giving an explicit counterexample. For a discussion of the history of the problem, and a presentation of related results, see the book of *Anosov and Bolibruch* [2]. Very recently, new results have been obtained for the same problem, but lower triangular monodromy matrices, by *Vandamme* [273], based on earlier work of *Bolibruch* [60].

12.4 Birkhoff's Reduction Problem

Let a system of ordinary differential equations of the form (3.1) (p. 37) be given. One classical question concerning such systems is that of the behavior of its solutions as $z \to \infty$. Since analytic transformations essentially leave this behavior unchanged, *Birkhoff* [53, 56] in 1913 suggested the following approach:

- Within an equivalence class of such systems, with respect to analytic transformations, determine the system(s) that in some sense are the simplest, and then study their solutions near the point infinity. This approach is very much analogous to the question of normal forms of constant matrices with respect to similarity, leading to Jordan canonical form.

Birkhoff conjectured that *for every system* (3.1) *one can always find an analytic transformation* $x = T(z)\,y$, *such that the transformed system* $z\,y' = B(z)\,y$, $B(z) = T^{-1}(z)\,[A(z)\,T(z) - z\,T'(z)]$, *has a polynomial* $B(z)$ *as its coefficient matrix*. In his honor, we shall call every system with polynomial coefficient matrix *a system in Birkhoff standard form*.

Birkhoff himself showed in [54] that the answer to his question is positive under the additional assumption that some monodromy matrix, around the point infinity, of (3.1) is diagonalizable, but seemed to believe that the same would hold in general. However, in 1959 *Gantmacher* [105] and *Masani* [188] independently presented examples of systems (3.1), in the smallest nontrivial dimension of $\nu = 2$, for which no such transformation exists. These counterexamples had triangular coefficient matrices, hence the following harder problem arose:

- Calling (3.1) *reducible* if an analytic transformation exists for which the transformed system is lower triangularly blocked, with square diagonal blocks of arbitrary dimensions, is it so that every *irreducible* system can be analytically transformed to Birkhoff standard form?

This question was answered positively, first for dimension $\nu = 2$ by *Jurkat, Lutz, and Peyerimhoff* [147], then for $\nu = 3$ in [15], and finally for any dimension by *Bolibruch* [61, 62, 64].

While the problem stated above concerns linear systems of ODE, all the attempts on proving it for various special cases are based on a general result on factorization of holomorphic matrices which was independently obtained by *Hilbert* [119] and *Plemelj* [220], as well as *Birkhoff* [56]:

> *Suppose that we are given a $\nu \times \nu$ matrix function $S(z)$, holomorphic for $|z| > \rho$, for some $\rho \geq 0$, whose determinant does not vanish there. Then*
>
> $$S(z) = T(z)\,E(z)\,z^K, \tag{12.1}$$
>
> *with an analytic transformation $T(z)$, an entire matrix function $E(z)$ whose determinant does not vanish for any $z \in \mathbb{C}$, and a diagonal matrix of integers $K = \operatorname{diag}[k_1, \ldots, k_\nu]$.*

For $\nu = 1$ one can easily show this through an additive decomposition of $\log S(z)$; however, the proof for $\nu \geq 2$ is much more involved and shall not be given here. For a very readable presentation of the basic ideas of Birkhoff's proof, see *Sibuya* [251].

The above factorization result applies to systems of ODE as follows: Let $X(z)$ be a fundamental solution of (3.1) with monodromy matrix M, so that $S(z) = X(z)\,z^{-M}$ is single-valued for $|z| > \rho$. Then $\det S(z)$ cannot vanish for these z, because of Proposition 1 (p. 6). With $T(z)$ as in (12.1), the transformation $x = T(z)\,\tilde{x}$ takes (3.1) into a system $z\tilde{x}' = B(z)\,\tilde{x}$, of the same Poincaré rank r, and having the fundamental solution $\tilde{X}(z) = E(z)\,z^K\,z^M$. So

$$B(z) = z\tilde{X}'(z)\,\tilde{X}^{-1}(z) = [zE'(z) + E(z)\,\{K + z^K M\,z^{-K}\}]\,E^{-1}(z),$$

showing that $B(z)$ is holomorphic and single-valued in $\mathbb{C}$, except for the origin. Moreover, we see from the form of $\tilde{X}$ that the origin is a *regular-singular point of the new system, and will be of first kind if $z^K M\,z^{-K}$ is a polynomial. This certainly holds whenever M is diagonal.* Hence we have proven the following result:

Theorem 60 *Every system* (3.1) *is analytically equivalent to one being singular only at infinity and at the origin, with the origin being regular-singular. In case* (3.1) *has a fundamental solution with diagonal monodromy matrix, then there exists an analytically equivalent system in Birkhoff standard form.*

In order to obtain Bolibruch's result on irreducible systems, we show a somewhat different factorization result:

Lemma 23 *Suppose that we are given a $\nu \times \nu$ matrix function $S(z)$, holomorphic for $|z| > \rho$, for some $\rho \geq 0$, whose determinant does not vanish there. Then*

$$S(z) = T(z)\, z^K\, E(z),$$

with an analytic transformation $T(z)$, an entire matrix function $E(z)$ whose determinant does not vanish for any $z \in \mathbb{C}$, and a diagonal matrix of integers $K = \operatorname{diag}[k_1, \ldots, k_\nu]$ satisfying $k_1 \geq k_2 \geq \ldots \geq k_\nu$.

Proof: For the proof, we factor $S(z)$ as in (12.1), and let $F(z) = E(z)\, z^K$. Then $\det F(z) = e(z)\, z^k$, $k = k_1 + \ldots + k_\nu$, $e(z) = \det E(z)$, hence $e(z) \neq 0$ for every $z \in \mathbb{C}$. The rows of $F(z)$ can be written as $f_j = \tilde{e}_j(z)\, z^{\tilde{k}_j}$, with $\tilde{k}_j \in \mathbb{Z}$, $\tilde{e}_j(z)$ holomorphic at the origin, and $\tilde{e}_j(0) \neq 0$. Without loss of generality, we may assume that $\tilde{k}_j$ are weakly decreasing with respect to j, since otherwise, we may permute the rows of $F(z)$ and columns of $T(z)$ accordingly. Note $F(z) = z^{\tilde{K}}\, \tilde{E}(z)$, with $\tilde{E}(z)$ having rows $\tilde{e}_j(z)$. Hence, $\det F(z) = z^{\tilde{k}}\, \tilde{e}(z)$, and $\tilde{k} \leq k$, $\tilde{e}(z) \neq 0$ for every z except possibly $z = 0$. Moreover, we have $\tilde{k} = k$ if and only if $\tilde{e}(0) \neq 0$, in which case the proof is completed. Suppose $\tilde{k} < k$. This occurs if and only if the rows of $\tilde{E}(0)$ are linearly dependent; however, note that no row vanishes, owing to the choice of $\tilde{k}_j$. In this situation, we choose $j \geq 2$ minimally, so that the jth row of $\tilde{E}(0)$ is linearly dependent on the earlier ones. We now add multiples of the ℓth row of $F(z)$ to the jth one, for $1 \leq \ell \leq j - 1$, the factor used being a constant times $z^{\tilde{k}_j - \tilde{k}_\ell}$. This operation is nothing but multiplication from the left with a special analytic transformation. Choosing the constants properly, we can achieve that the new matrix, which for simplicity is again denoted by $F(z)$, has the form $z^{\tilde{K}}\, \tilde{E}(z)$, where now the jth row of $\tilde{E}(0)$ vanishes. Consequently, we factor $F(z)$ as $z^{\bar{K}}\, \bar{E}(z)$, with $\bar{K}$, $\bar{F}(z)$ as above, but $\bar{k}_1 + \ldots + \bar{k}_\nu > \tilde{k}$. Repeating this finitely many times, the proof can be completed. □

We now show Bolibruch's result:

Theorem 61 *Every irreducible system* (3.1) *is analytically equaivalent to one in Birkhoff standard form.*

Proof: Choose a fundamental solution of (3.1) of the form $X(z) = S(z)\, z^J$, with a monodromy matrix J in lower triangular Jordan form, and $S(z)$ single-valued and holomorphic for $|z| > \rho$. Let $D = \operatorname{diag}[d_1, \ldots, d_\nu]$ have integer diagonal elements with $d_j - d_{j+1} > r\,(\nu - 1)$, and apply the above lemma to $S(z)\, z^{-D}$ to obtain $X(z) = T(z)\, z^K\, E(z)\, z^D\, z^J$. For $B(z) = T^{-1}(z)\,[A(z)\, T(z) - z\, T'(z)]$, the system $z\, y' = B(z)\, y$ then has the fundamental solution $Y(z) = z^K\, E(z)\, z^D\, z^J$. This implies

$$z^{-K}\, B(z)\, z^K = K + [z\, E'(z) + E(z)\, (D + z^D\, J\, z^{-D})]\, E^{-1}(z).$$

Because of J lower triangular and d_j decreasing, we find the right-hand side to be holomorphic at the origin. If there was a j with $k_j - k_{j+1} > r$, then the fact that $z\,y' = B(z)\,y$ has Poincaré rank r would imply $B(z)$ triangularly blocked. This, however, would contradict the irreducibility of (3.1). Hence, $k_j - k_{j+1} \le r$ for $1 \le j \le \nu - 1$ follows. Using Exercise 2, we conclude $Y(z) = \bar{T}(z)\,\tilde{Y}(z)$, with $\tilde{Y}(z) = \bar{E}(z)\,z^{\tilde{K}+D}\,z^J$, where $\bar{E}^{\pm}(z)$ are entire, and $\tilde{K}$ is equal to K but for a permutation of its diagonal elements. Owing to our choice of D, we find that the diagonal elements of $\tilde{K} + D$ are decreasing, so that $z\,\tilde{Y}'(z)\,\tilde{Y}^{-1}(z)$ is holomorphic at the origin, i.e., in fact, is a polynomial. $\square$

As we pointed out above, the problem of Birkhoff standard form arose in the study of behavior of solutions of (3.1) for $z \to \infty$. This behavior is not too drastically altered even when using meromorphic transformations instead of analytic ones. So it is natural to ask whether transformation to Birkhoff standard form is always possible using meromorphic transformations. The answer to this question is positive, once we allow the transformation to increase the Poincaré rank of the system. However, it is more natural to restrict to meromorphic transformations leaving the rank the same. For dimensions $\nu = 2$, resp. $\nu = 3$, *Jurkat, Lutz, and Peyerimhoff* [147], resp. *Balser* [14], have shown the answer to this question to be positive. For general dimensions, but under the additional assumption of the leading matrix A_0 of (3.1) having distinct eigenvalues, *Turrittin* [271] also obtained a positive answer, but in general this problem is still open. For numerous sufficient conditions under which a positive answer is known, see *Balser and Bolibruch* [30].

Exercises: Let $E(z)$ be an entire $\nu \times \nu$ matrix function with $\det E(0) \ne 0$, and let $k_j \in \mathbb{Z}$, $k_1 \ge \ldots \ge k_\nu$ be given.

1. Show $E(z) = P(z)\,\tilde{E}(z)\,R$, with:
   - $P(z) = [p_{j\ell}]$ is a lower triangular matrix with $p_{jj}(z) \equiv 1$, and $p_{j\ell}(z)$ polynomials of degree at most $k_\ell - k_j$, for $1 \le \ell < j \le \nu$;
   - $\tilde{E}(z) = [\tilde{e}_{j\ell}(z)]$ is entire, with $\tilde{e}_{j\ell}(z)$ vanishing at the origin at least of order $k_\ell - k_j + 1$ for $1 \le \ell < j \le \nu$;
   - R is a permutation matrix.
2. With $K = \operatorname{diag}[k_1, \ldots, k_\nu]$, show $z^K\,E(z) = \bar{T}(z)\,\bar{E}(z)\,z^{\tilde{K}}$, with an analytic transformation $\bar{T}(z)$, $\bar{E}(z)$ entire, $\det \bar{E}(z) \ne 0$ everywhere, and $\tilde{K}$ a diagonal matrix of integers, differing from K only by permutation of its diagonal elements.

12.5 Central Connection Problems

Generally speaking, a *connection problem* is concerned with two fundamental solutions $X_1(z)$, $X_2(z)$ of the same system of ODE, say, of the form (1.1) (p. 2). The solutions may be given in terms of power series, or integrals, converging in regions $G_1, G_2 \subset G$, and usually have certain natural properties there. According to Theorem 1 (p. 4), both solutions $X_j(z)$ can be holomorphically continued into all of G, and then are related as $X_1(z) = X_2(z)\,\Omega$, with a unique invertible constant matrix Ω. This matrix then is the corresponding *connection matrix*, and its computation is referred to as the *connection problem*. For example, the system may be of the form (3.1) (p. 37), and the fundamental solutions can be two consecutive highest-level normal solutions $X_j(z)$ and $X_{j-1}(z)$, which were introduced in Section 9.1 and are characterized through their Gevrey asymptotic in the corresponding sectors S_j, S_{j-1}. In this setting, the connection problem is the same as the computation of the corresponding Stokes multiplier of highest level and has been discussed in Chapter 9. While such problems sometimes are called *lateral connection problems*, we shall here be concerned with another type: Consider a system that is singular at two points z_1, z_2. Assume z_1 to be of first kind, or at least regular-singular; then a fundamental solution $X_1(z)$ can be obtained as $X_1(z) = \sum_0^\infty S_m\,(z-z_1)^{m\,I+M}$, $|z-z_1| < \rho$, with a monodromy matrix M and matrix coefficients S_n that at least theoretically can be computed from the system. Then the problem arises of how this fundamental solution behaves as we approach the other singularity z_2. *To study this behavior is what we call the central connection problem.*

Such problems, with z_2 also being regular-singular, have been treated, e.g., by *Schäfke* [238], resp. *Schäfke and Schmidt* [241, 253]. Here, we shall instead assume that the point z_2 is irregular-singular. This situation, under various additional assumptions, has been investigated by, among others, *Newell* [202], *Kazarinoff and Kelvey* [150], *Knobloch* [152], *Wasow* [280], *Kohno* [156–159], *Wyrwich* [285, 286], *Okubo* [206], *Naundorf* [198–200], *Bakken* [6, 7], *Jurkat* [143], *Schäfke* [237, 239, 240], *Paris and Wood* [216–218], *Paris* [215], *Lutz* [175], *Kovalevski* [163, 164], *Balser, Jurkat, and Lutz* [37, 41], *Yokoyama* [287–289], *Balser* [13], *Lutz and Schäfke* [178], *Okubo, Takano, and Yoshida* [208], *Sibuya* [251], and *Reuter* [231, 232]. For numerical investigations and applications to problems in physics, see a recent article by *Lay and Slavyanov* [166] and the literature quoted there.

Here, we shall study the central connection problem in the following general setting: We shall consider a system (3.1) (p. 37), *where $A(z)$ is a rational matrix function with poles only at the origin and infinity.* Moreover, we shall make the following additional assumptions upon the nature of the singularities at the origin resp. infinity:

1. The origin is supposed to be a *regular-singular point* of the system, but may not be a singularity of first kind. Moreover, assume that a

fundamental solution of the form $X(z) = S(z)\, z^M$, $S(z) = \sum_0^\infty S_n\, z^n$, has been computed. Note that, owing to the absence of other finite singular points, the power series automatically has an infinite radius of convergence; hence $S(z)$ is an entire function, and $\det S(z) \neq 0$ for every $z \neq 0$.

2. Infinity is supposed to be an *essentially irregular singularity* with an HLFFS $(\hat{F}(z), Y(z))$, satisying the assumptions in Section 9.3. Note that these restrictions are without loss of generality, since some easy normalizing transformations can be used to make them hold. Also, recall from Section 9.3 the definition of the associated functions and their behavior in the cut plane $\mathbb{C}_d$, for every nonsingular direction d.

Under these assumptions, let $X_j(z) = F_j(z)\, Y(z)$ be the normal solutions of highest level. Then there exist unique invertible matrices Ω_j, so that $X(z) = X_j(z)\, \Omega_j$, $j \in \mathbb{Z}$. What we are going to show is how the *central connection matrices* Ω_j can be computed via an analysis of some functions $\Psi(u; s; k)$, corresesopnding to the HLNS via Laplace transform.

Let $k \in \mathbb{Z}$, and recall the definition of $j^*(k)$ from p. 147. For α with $d_{j^*(k)-1} - \pi/(2r) < \alpha < d_{j^*(k)} + \pi/(2r)$, consider the integral

$$\Psi(u; s; k) = \frac{r}{2\pi i} \int_0^{\infty(\alpha)} z^{s-1} X(z)\, \mathrm{e}^{z^r u}\, dz.$$

The assumptions made above imply that $X(z)$ is of moderate growth at the origin, and of exponential growth at most r in arbitrary sectors at infinity. Therefore, the integral converges absolutely and locally uniformly for Re s sufficiently large and u in a sectorial region (near infinity) of opening π and bisecting direction $\pi - r\,\alpha$. We have $X(z) = S(z)\, z^M$, with an entire function $S(z)$ of exponential growth at most r, so that it is justified to termwise integrate the power series expansion of $S(z)$. Making a change of variable $z^r\, u = \mathrm{e}^{i\pi}\, x$ in the above integral, we obtain for s and u as above

$$\Psi(u; s; k) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} S_n\, \Gamma([(n+s)\, I + M]/r)\, (\mathrm{e}^{i\pi}/u)^{[(n+s)\, I + M]/r}, \quad (12.2)$$

where the *matrix Gamma function* $\Gamma(A)$ here is defined by the integral

$$\Gamma(A) = \int_0^\infty x^{A-I}\, e^{-x}\, dx.$$

Observe that this integral converges absolutely for every matrix A whose eigenvalues all have positive real parts. Integrating by parts, we can show $A\,\Gamma(A) = \Gamma(I+A)$. Using this, it is possible to extend the definition of $\Gamma(A)$ to matrices having no eigenvalue equal to a nonpositive integer. Therefore, the expansion (12.2) can serve as holomorphic continuation of $\Psi(u; s; k)$, with respect to s, to become a meromorphic function of s with poles at the points

$$s + \mu = -j, \qquad j \in \mathbb{N}_0.$$

This will, however, not be needed here.

For the auxiliary functions, defined on p. 147, we may choose $z_0 = 0$, whenever Re s is large enough. Doing so, we find

$$\Psi(u; s; k) = \sum_{m=1}^{\mu} \Phi_m^*(u; s; k)\, \Omega_m^{(k)},$$

where $\Omega_m(k)$ denotes the mth row of blocks of $\Omega_{j^*(k)}$, when this matrix is blocked of type $(s_1, \ldots, s_\mu)$. This shows that $\Psi(u; s; k)$ is holomorphic in $\mathbb{C}_d$. For its singular behavior at the points u_n, recall that $\Phi_m^*(u; s; k) = \mathrm{hol}(u - u_n)$ for $m \neq n$, while we have shown in the proof of Theorem 46 (p. 149) that $\sum_{\ell=0}^{r-1} \Phi_n(u; s; k+\ell) = \Phi_n^*(u; s; k)\,(I - \mathrm{e}^{-2\pi i\,(sI+L_n)}) + \mathrm{hol}(u - u_n)$. This shows:

Theorem 62 *Under the assumptions made above, we have for every s with* Re s *large and so that $(I - \mathrm{e}^{2\pi i\,(sI+L_n)})^{-1}$ exists,*

$$\Psi_m(u; s; k) = \left[\sum_{\ell=0}^{r-1} \Phi_n(u; s; k+\ell)\right] (I - \mathrm{e}^{-2\pi i\,(sI+L_n)})^{-1}\, \Omega_n^{(k)} + \mathrm{hol}(u - u_n),$$

for every $n = 1, \ldots, \mu$.

This identity shows that the central connenction problem can theoretically be solved as follows: *Compute the matrix $\Psi(u; s; k)$, either by its integral representation or the convergent power series* (12.2)*, for u and* Re s *sufficiently large. Then, continue the function with respect to u to the singularities u_n, and there use the above identity to compute $\Omega_n^{(k)}$. Doing this for every n allows to compute the matrix $\Omega_{j^*(k)}$, linking the fundamental solution $X(z)$ to the corresponding normal solution of highest level.*

Without going into detail, we mention that given two matrices $\Omega_{j^*(k)}$ and $\Omega_{j^*(k+1)}$, one can compute the Stokes multipliers V_j, for $j^*(k+1)+1 \leq j \leq j^*(k)$, by factoring $\Omega_{j^*(k+1)}\, \Omega_{j^*(k)}^{-1}$ as in the exercises in Section 9.2. Consequently, if we have computed $\Omega_{j^*(k+\ell)}$, for $\ell = 0, \ldots, r-1$, then all Stokes' multipliers of highest level can be found explicitly. However, the knowledge of the Stokes multipliers is not sufficient to find the central connection matrices: Assume that we had chosen $X(z)$ so that $\exp[2\pi i\, M]$ were in Jordan normal form. Moreover, assume all Stokes' multipliers of highest level are known. Then the monodromy factor $\exp[2\pi i\, M_j]$ for $X_j(z)$ is given by (9.5). So by continuation of the relation $X(z) = X_j(z)\, \Omega_j$ about infinity we obtain $\Omega_j \exp[2\pi i\, M] = \exp[2\pi i\, M_j]\, \Omega_j$. This then shows that Ω_j is some matrix which transforms $\exp[2\pi i\, M_j]$ into Jordan form. *However, such a matrix is not uniquely determined.* In the generic situation where

$\exp[2\pi i\, M_j]$ is diagonalizable, Ω_j is determined up to a right-hand diagonal matrix factor, and this is exactly the degree of freedom we have in choosing a fundamental solution $X(z)$ consisting of Floquet solutions. Therefore, the knowledge of the Stokes multipliers alone does not determine Ω_j.

13

Applications in Other Areas, and Computer Algebra

In this chapter we shall briefly describe applications of the theory of multisummability to formal power series solutions of equations other than linear ODE. The efforts to explore such applications are far from being complete and shall provide an excellent chance for future research. In a final section we then mention recent results on finding formal solutions for linear ODE with help of computer algebra.

Suppose that we are given some class of functional equations having solutions that are analytic functions in one or several variables. Roughly speaking, we shall then address the following two questions:

- Does such a functional equation admit *formal solutions* that, aside from elementary functions such as exponentials, logarithms, general powers, etc., involve formal power series in one or several variables? Can one, perhaps, even find a family of formal solutions that is *complete* in some sense?

- Given such a formal solution, are the formal power series occurring, if not already convergent, summable in some sense or another, and if so, are the functions obtained by replacing the formal series with their sums then solutions of the same functional equation?

Two general comments should be made beforehand: First, it is not at all clear whether one should in all cases consider formal solutions involving formal *power series* – e.g., one could instead consider formal *factorial series.* One one hand, each formal power series can be formally rewritten as a formal factorial series and vice versa, so both approaches may seem equivalent. On the other hand, however, it may be easier to find the coefficients for a factorial series solution directly from the underlying functional equation, and the question of summation for a factorial series may have an easier answer than for the corresponding power series. Since we have only discussed summation of formal power series, we shall here restrict to that case, but mention a paper by *Barkatou and Duval* [48], concerning summation of formal factorial series. The second comment we wish to make concerns the question of summation of formal power series in several variables: In the situations we are going to discuss we shall always treat all but one variable as (temporarily fixed) parameters, thus leaving us with a series in the remaining variable, with coefficients in some Banach space. It is for this reason that we have developed the theory of multisummability in such a relatively general setting. While this approach is successful in some situations, there are other cases indicating that one should better look for a summation method that treats all variables simultaneously, but so far nobody has found such a method!

13.1 Nonlinear Systems of ODE

Throughout this section, we shall be concerned with *nonlinear systems* of the following form:

$$z^{-r+1}x' = g(z,x), \tag{13.1}$$

where r, the *Poincaré rank*, is a nonnegative integer, $x = (x_1,\dots,x_\nu)^T$, $\nu \geq 1$, and $g(z,x) = (g_1(z,x),\dots,g_\nu(z,x))^T$ is a vector of power series in $x_1,\dots,x_n$. Let $p = (p_1,\dots,p_\nu)^T$ be a *multi-index*, i.e., all p_j are nonnegative integers, and define $x^p = x_1^{p_1}\cdot\ldots\cdot x_\nu^{p_\nu}$. Then such a power series can be written as $g_j(z,x) = g_j(z,x_1,\dots,x_n) = \sum_{p\geq 0}^{\infty} g_{j,p}(z)\,x^p$. The coefficients $g_{j,p}(z)$ are assumed to be given by power series in z^{-1}, say, $g_{j,p}(z) = \sum_{m=0}^{\infty} g_{j,p,m}\,z^{-m}$. *Throughout, it will be assumed that all these series converge for* $|z| > \rho$, *with some* $\rho \geq 0$ *independent of* p, *while the series for* $g_j(z,x)$, *for every such* z, *converges in the ball* $\|x\| < \rho$.

As is common for multi-indices, we write $|p| = p_1 + \ldots + p_\nu$. If it so happens that $g_{j,p}(z) \equiv 0$ whenever $|p| \geq 2$, for every j, then (13.1) obviously becomes an inhomogeneous linear system of ODE, whose corresponding homogeneous system is as in (3.1) (p. 37). If $g_{j,0}(z) \equiv 0$ for every j, then (13.1) obviously has the solution $x(z) \equiv 0$, and we then say that (13.1) *is a homogeneous nonlinear system.*

It is shown in [24] that homogeneous nonlinear systems have a formal solution $\hat{x}(z)$, sharing many of the properties of FFS in the linear case:

1. The formal solution $\hat{x}(z)$ is a formal power series $\hat{x}(z) = \hat{x}(z,c) = \sum_{|p|\geq 0} \hat{x}_p(z)\,c^p$ in *parameters* $c_1,\dots,c_\nu$. Its coefficients $\hat{x}_p(z)$ are finite sums of expressions of the form $\hat{f}(z)\,z^\lambda \log^k z\ \exp[p(z)]$, with a

formal power series $\hat{f}$ in z^{-1}, a complex constant λ, a nonnegative integer k, and a polynomial p in some root of z. In case of a linear system, the coefficients $\hat{x}_p(z)$ are zero for $|p| \geq 2$, so that then $\hat{x}(z, c)$ is a linear function of $c_1, \ldots, c_\nu$ and corresponds to a FFS.

2. The proof of existence of $\hat{x}(z)$ follows very much the same steps as in the linear case: One introduces nonlinear analytic, resp. meromorphic, transformations and shows that by means of finitely many such transformations one can, step by step, simplify the system in some clear sense so that in the end one can "solve" it explicitly. For details, see [22, 24].

3. The formal power series occurring in the coefficients $\hat{x}_p(z)$ all are multisummable, as is shown in [26].

Despite all the analogies to the linear situation, there are two new difficulties for nonlinear systems: For one, it is not clear whether the formal solution $\hat{x}(z, c)$ is *complete* in the sense that every other formal expression solving (13.1) can be obtained from $\hat{x}(z, c)$ by a suitable choice of the parameter vector c. Moreover, suppose that all the formal power series in $\hat{x}(z, c)$ are replaced by their multisums, so that we obtain a formal power series in the parameter vector c, with coefficients which are holomorphic functions in some sectorial region G at infinity. It is a well-known fact, called *the small denominator phenomenon*, that in general this series diverges. Sufficient conditions for convergence are known; see, e.g., the papers of *Iwano* [139, 140] and the literature quoted there. For an analysis of the nonlinear Stokes phenomenon under a nonresonance condition, compare *Costin* [83]. In general, however, it is still open how this series is to be interpreted.

A related but simpler problem for nonlinear systems is as follows: Suppose that (13.1) has a solution in the form of a power series in z^{-1}, then is this series multisummable? By now, there are three proofs for the answer being positive, using quite different methods: *Braaksma* [70] investigated the nonlinear integral equations of the various levels which correspond to (13.1) via Borel transform. *Ramis and Sibuya* [230] used cohomological methods, while in [21] a more direct approach is taken, very much like the proof of *Picard-Lindelöf's existence and uniqueness theorem.*

13.2 Difference Equations

While multisummability is a very appropriate tool for linear and nonlinear ODE, things are more complicated for difference equations, as we now shall briefly explain. For a more complete presentation of the theory of holomorphic difference equations, see the recent book of *van der Put* *and Singer* [224]. In his Ph.D. thesis, *Faber* [102] considers extensions to differential-difference equations and more general functional equations that we do not wish to include here. For simplicity we restrict to *linear systems* of difference equations, although much of what we say is known to extend to the nonlinear situation: Let us consider a system of difference equations of the form

$$x(z+1) = A(z)\, x(z),$$

where the coefficient matrix is exactly as in (3.1), i.e., is holomorphic for $|z| > \rho$ and has at worst a pole at infinity. The first, trivial although important, observation is that the order of the pole of $A(z)$ does not play the same crucial role as for systems of ODE: If we set $x(z) = \Gamma(z)\, \tilde{x}(z)$, then the functional equation of the Gamma function implies that $\tilde{x}(z)$ satisfies a difference equation exactly like before, but with $z^{-1}A(z)$ as its coefficient matrix. Thus we learn by repeated application of this argument that we might restrict ourselves to systems where $A(z)$ is *holomorphic at infinity, and* $A(\infty) \neq 0$, but for what we shall have to say, this will be not essential. However, to avoid degeneracies, we shall assume that $\det A(z) \not\equiv 0$, so that $x(z+1) = A(z)\, x(z)$ can be solved for $x(z)$, obtaining an equivalent difference equation in the "backward direction."

Under the assumptions made above, the formal theory of such a difference equation is well established [80, 81, 92, 133, 134, 223, 269]: Every such system has a formal fundamental solution of the form

$$\hat{X}(z) = \hat{F}(z)\, z^{\Lambda z}\, e^{Q(z)}\, z^{L},$$

with:

- a formal q-meromorphic transformation $\hat{F}(z)$, for some $q \in \mathbb{N}$,
- a diagonal matrix Λ of rational numbers with denominators equal to the same number q,
- a diagonal matrix $Q(z)$ of polynomials in $z^{1/q}$ without constant term and of degree strictly less than q, and
- a constant matrix L in Jordan canonical form, commuting with $Q(z)$ and Λ.

Hence, the main difference between formal solutions of difference and differential equations is the occurrence of the term $z^{\Lambda z}$. Indeed, if Λ vanishes or is a multiple of the identity matrix, then *Braaksma and Faber* [71] have shown that $\hat{F}(z)$ is $(k_1, \ldots, k_p)$-summable in *all but countably many directions*, with levels $1 = k_1 > \ldots > k_p > 0$ that are determined by $Q(z)$ in exactly the same fashion as for ODE. If Λ contains several distinct entries on the diagonal, however, things get considerably more complicated because of the occurrence of a new *level*, commonly denoted as *level* 1^+. A simple but very instructive example showing this phenomenon may be found in Faber's thesis [102]. Roughly speaking, this example shows that in presence of level 1^+ one can no longer restrict to using Laplace transform integrating along straight lines, but has to allow other paths of integration. For a general presentation, see Chapter 3 of the said thesis, containing joint work of *Braaksma, Faber, and Immink.*

13.3 Singular Perturbations

Let us consider a system of ODE of the form

$$\varepsilon^\sigma\, x' = g(z, x, \varepsilon),$$

where $g(z, x, \varepsilon)$ is as in Section 13.1, but additionally depends on a parameter ε, and σ is a natural number. Analysis of the dependence of solutions on ε is referred to as *a singular perturbation problem.* Under suitable assumptions on the right-hand side, such a system will have a *formal solution* $\hat{x}(z, \varepsilon) = \sum_{n=0}^\infty x_n(z)\, \varepsilon^n$, with coefficients $x_n(z)$ given by *differential recursion relations.* In general, the series is divergent, and classically one has tried to show existence of solutions of the above system that are asymptotic to $\hat{x}(z, \varepsilon)$ when $\varepsilon \to 0$ in some sectorial region. Very recently, one has begun to investigate Gevrey properties of $\hat{x}$, or discuss its (multi-)summability. Of the recent articles containing results in this direction, we mention *Wallet* [278, 279], *Canalis-Durant* [76], and *Canalis-Durant, Ramis, Schäfke, and Sibuya* [77].

Since here we meet power series whose coefficients are functions of another variable, the situation is very much analogous to that of formal solutions of partial differential equations, which we are investigating in the following section. Here, we shall briefly look at a very simple example, already discussed by *Ecalle*:

The inhomogeneous equation $\varepsilon\, x' = x - f(z)$ has the formal solution $\hat{x}(z, \varepsilon) = \sum_0^\infty \varepsilon^n\, f^{(n)}(z)$, for arbitrary $f(z)$ which we assume holomorphic near the origin, say, for $|z| < \rho$. According to *Cauchy's formula*, the coefficients $f^{(n)}(z)$ grow roughly like $n!$, so it is natural to study 1-summability of this series. The formal Borel transformation of order $k = 1$ of this series equals the power series expansion of $f(z + \varepsilon)$ about the point z. Consequently, holomorphic continuation of $\hat{\mathcal{B}}_1 \hat{x}$ is equivalent to that of f. *Hence, the series $\hat{x}$ is 1-summable in a direction d if and only if the function f admits holomorphic continuation into a sector $S(d, \delta)$, for some $\delta > 0$, and is of exponential growth at most 1 there.* More generally, one can prove a result on the multisummability of $\hat{x}$ that is completely analogous to Theorem 63 (below) for the heat equation. In any case, multisummability of $\hat{x}$ is always linked to explicit conditions on the function $f(z)$. This indicates that for power series of several variables, the notion of multisummability that has been developed in this book is not general enough. This is due to the fact that here we can only treat one variable at a time, while the remaining ones are treated as parameters.

13.4 Partial Differential Equations

There is a classical theory discussing *convergence* of power series solutions for Cauchy problems for certain classes of partial differential equations (PDE). Recently, efforts have been made to show that formal power series arising as solutions of such problems have a certain Gevrey order; for such results, see [108, 190–194, 213] and the literature cited there. There are also results, e.g., by *Ouchy* [214], showing that such formal solutions are asymptotic representations of proper solutions of the underlying equation. Not much is known so far about summability of formal solutions of such problems: *Lutz, Miyake, and Schäfke* [176] obtained a first result for the *complex heat equation*, which has been generalized in [27]. Very recent work by *Balser and Miyake* [42] treats an even more general case, showing that the results obtained are not so much dependent on having a formal solution of a partial differential equation, but carry over to series whose coefficients are given by certain differential recursions.

Here, we briefly investigate the following situation: Given a function $\varphi(z)$, analytic in some G containing the origin, try to find another function $u(t,z)$ of two complex variables, in some subset of $\mathbb{C}^2$ near the origin, such that

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial z^2} u, \qquad u(0,z) = \varphi(z). \tag{13.2}$$

Obviously, (13.2) has the formal solution $\hat{u}(t,z) = \sum_{n=0}^{\infty} t^n\, \varphi^{(2n)}(z)/n!$ In an arbitrary compact subset $K \subset G$, one can show, using Cauchy's formula for derivatives, that $\|\varphi^{(2n)}\|_K = \sup_{z\in K} |\varphi^{(2n)}(z)| \le c^n\, \Gamma(1+2n)$, for sufficiently large c. Hence $\hat{u}$ can be regarded as a formal power series in t with coefficients in the Banach space $\mathbb{E}_K$ of functions that are bounded on K and analytic in its interior. The above estimate implies $\hat{u} \in \mathbb{E}_K[[t]]_1$. Thus, it is natural to ask for 1-summability of $\hat{u}$. If φ happens to be entire, however, the above estimate upon its derivatives may be improved, and then $\hat{u}$ may be summable of another type. Indeed, one can show that for arbitrary $k_1 > \ldots > k_q > 1/2$, under certain explicit conditions, best expressed in the coefficients of the power series expansion $\varphi(z) = \sum \varphi_n\, z^n$, one even has $(k_1,\ldots,k_q)$-summability of $\hat{x}$. In detail, the following holds [27]:

Theorem 63 *Let $\varphi(z) = \sum_0^\infty \varphi_n\, z^n$ for $|z| < \rho$, and define $\hat{u}$ as above, resp. for $0 \le j \le 2$:*

$$\hat{\psi}_j(t) = \sum_0^\infty \varphi_{2n+j} \frac{(2n+j)!}{n!}\, t^n, \;\; \hat{\psi}^{\pm}(t) = \sum_0^\infty \varphi_n\, \Gamma(1+n/2)\, (\pm t)^n.$$

Given any type of multisummability $k = (k_1, \ldots, k_\nu)$ with $k_\nu \geq 1$, and any multidirection $d = (d_1, \ldots, d_\nu)$, admissible with respect to k, let $2k = (2k_1, \ldots, 2k_\nu)$, $d/2 = (d_1/2, \ldots, d_\nu/2)$. Then the following statements are equivalent:

(a) *For every $K \subset D(0, \rho)$, $\hat{u} \in \mathbb{E}_K\{t\}_{k,d}$.*

(b) *For $0 \leq j \leq 1$, $\hat{\psi}_j \in \mathbb{C}\{t\}_{k,d}$.*

(c) $\hat{\psi}^{\pm} \in \mathbb{C}\{t\}_{2k,d/2}$.

Proof: Assume (a). For $K_1 \subset D(0, \rho)$, let $K_2 \subset D(0, \rho)$ be so that K_1 is contained in the interior of K_2. Then Cauchy's formula for the derivative implies that $D = d/dz$ is a bounded linear operator from $\mathbb{E}_{K_2}$ into $\mathbb{E}_{K_1}$. From Theorem 52 (p. 171) we conclude that $\hat{u} \in \mathbb{E}_{K_2}\{t\}_{k,d}$ implies $\hat{u}_z(t, z) = \sum_0^\infty \varphi^{(2n+1)}(z)/n! \in \mathbb{E}_{K_1}\{t\}_{k,d}$. Taking $z = 0$ and using Exercise 4 on p. 172, this implies (b). From Theorem 51 (p. 166) we obtain that (b) is equivalent to $\hat{\psi}_j(t^2)$ being in $\mathbb{C}\,\{t\}_{2k,d/2}$. Applying Theorem 50 (p. 164) and Exercise 4 on p. 109, one can then show equivalence with $\sum_0^\infty \Gamma(1+n)\, \varphi_{2n}\, t^{2n}$ and $\sum_0^\infty \Gamma(3/2+n)\, \varphi_{2n+1}\, t^{2n}$ being in $\mathbb{C}\,\{t\}_{2k,d/2}$. This in turn is equivalent to (c). This leaves to conclude (a) from either (b) or (c): To do so, first observe that, as a consequence of Theorem 50 (p. 164), it suffices to consider the case $\nu = 1$. In this situation, verify that $\hat{u}(t, z) = \sum_{m=0}^\infty z^m\, \hat{\psi}_m(t)$, with $\hat{\psi}_0(t)$, $\hat{\psi}_1(t)$ as in (b), and $\hat{\psi}_m(t) = \hat{\psi}_j^{(\mu)}(t)/m!\,$, for $m = 2\mu + j$, $0 \leq j \leq 1$. So $\hat{\psi}_m \in \mathbb{C}\,\{t\}_{k,d}$, and $\psi_m(t) = (\mathcal{S}_{k,d}\, \hat{\psi}_m)(t) = \psi_j^{(\mu)}(t)/m!$ in some sectorial region $G = G(d, \alpha)$, $\alpha > \pi/k$, independent of m. From Proposition 9 (p. 70) we obtain for every closed subsector $\bar{S} \subset G$ existence of constants $C, c > 0$ with $|\psi_m(t)| \leq C\, c^m\, \Gamma(1 + m\, s)/m!$, $t \in \bar{S}$, $m \geq 0$, for $s = (1 + 1/k)/2$. Owing to $k \geq 1$, the series $u(t, z) = \sum_{m=0}^\infty z^m\, \psi_m(t)$ converges uniformly (in two variables) for $|z| \leq \rho$, $\rho > 0$ sufficiently small and $t \in \bar{S}$[1]. So $u(t, z) \in \boldsymbol{H}(G, \mathbb{E}_K)$, for every K as in (a). Differentiating with respect to t, one can show

$$\partial_t^n\, u(t, z) = \sum_{\mu=0}^\infty \left[z^{2\mu} \frac{\psi_0^{(\mu+n)}(t)}{(2\mu)!} + z^{2\mu+1} \frac{\psi_1^{(\mu+n)}(t)}{(2\mu+1)!} \right] \to \varphi^{(2n)}(z),$$

as $t \to 0$. Hence we find that $u(t, z)$ has $\hat{u}(t, z)$ as its asymptotic expansion. To show that this expansion is of Gevrey order k, we use Proposition 9 (p. 70) again to obtain

$$|\partial_t^n\, u(t, z)| \quad \leq \quad C \sum_{m=0}^\infty (c\,|z|)^m\, \Gamma(1 + s(m+n))/m!$$

[1]In fact, if $k > 1$, i.e., $\varphi(z)$ entire, convergence takes place for every z.

$$= \quad C \int_0^\infty x^{sn} \exp[-x + c|z|\, x^s]\, dx.$$

The integral can be bounded by $\Gamma(1+sn)$ times some constant to the power n, completing the proof. □

It is worthwhile observing that for $t \neq 0$ the series $u(t,z)$ in fact converges for every z, and hence represents an entire function in z that may be seen to be of exponential size at most 2. This coincides with the classical result on the convergence of $\hat{u}(t,z)$ (see below). The formal series $\hat{\psi}_j(z)$ resp. $\hat{\psi}^{\pm}(z)$ are explicitly related to $\varphi(z)$, and using acceleration and Laplace operators, one can explicitly reformulate (a), (b) *in terms of transforms* of the function $\varphi(z)$. However, it is in general difficult to check (a) or (b) *directly* through investigation of $\varphi(z)$, except for the special case of $\nu = 1$, $k = k_1 = 1$. In this situation, the above theorem essentially coincides with the result obtained by *Lutz, Miyake, and Schäfke* [176]: Conditions (a) or (b) then are equivalent to $\varphi(z)$ admitting holomorphic continuation into small sectors bisected by rays $\arg z = d$ and $\arg z = d + \pi$, and being of exponential size not more than 2 there.

As follows from the proof of the above theorem, in case of $k_\nu > 1$ we have summability of $\hat{u}(t,z)$ for every z. For $k_\nu = 1$, however, summability takes place only in a disc whose radius in general is smaller than the radius of convergence of $\hat{\varphi}(z)$. Since k-summability in all directions is equivalent to convergence, we obtain as a corollary of the above theorem that convergence of $\hat{u}(t,z)$ is equivalent to the initial condition being an entire function of exponential size at most two. This, however, is a well-known classical result.

13.5 Computer Algebra Methods

While in principle the computation of a formal fundamental solution of a system (3.1) presents no problem, it can in practice become very tiresome, even for relatively small dimensions. Thus it may be good to know that there are computer algebra packages available that perform these computations. Others are being developed right now, aiming at even better performance and/or giving more detailed information on the structure of the formal fundamental solution. One should note, however, that one certainly cannot hope for expressing *all the terms* in such a FFS in closed form. However, it is possible to compute explicitly the exponential polynomials and the formal monodromy matrix, and even set up the recursion equations from which one then may obtain any finite number of power series coefficients. In the same sense, formal fundamental solutions of difference equations may be obtained with help of a computer, but we shall not discuss this here.

As we have seen previously, scalar equations of arbitrary order can be rewritten as systems. Vice versa, systems can, with help of a cyclic vector, be transformed into scalar equations (in Exercise 1.8 on p. 5 we have briefly discussed cyclic vectors near a regular point of an ODE, but one can also show existence of cyclic vectors in a neighborhood of a singular point [84]). So in principle, a computer algebra package written for systems may also be used for scalar equations, or vice versa, but there may be differences in performance.

Algorithms for the scalar case are usually based on the Newton polygon construction and Newton's algorithm [44, 123, 263]. Available programs in MAPLE are the package DIFFOP (see *van Hoeij* [122–124]), which can be down-loaded at

`http://klein.math.fsu.edu/ hoeij/maple.html`,

the ELISE package of *Dietrich* [88] at

`www.math2.rwth-aachen.de/~dietrich/elise.html`

and versions of the package DESIR developed at Grenoble in MAPLE and REDUCE.

Much work has been done at the last institute for the system case with the emphasis on the algorithmical efficient resolution [45–47, 49, 79, 117, 118]. The package ISOLDE recently developed by *Barkatou and Pflügel* in MAPLE contains several functions computing formal solutions and other applications for systems. It is, together with DESIR, available at

`http://www-lmc.imag.fr/CF/logiciel.html`.

14
Some Historical Remarks

Since the middle of the last century, mathematicians and physicists alike have observed that simple ODE may have solutions in terms of power series whose coefficients grow at such a rate that the series has a radius of convergence equal to zero. In fact, it became clear later that every linear meromorphic system, in the vicinity of a so-called *irregular singularity*, has a *formal fundamental solution* of a certain form. This "solution" can be relatively easily computed, in particular nowadays, where one can use computer algebra packages, but it will, in general, involve power series diverging everywhere. It was quite an achievement when it became clear that formal, i.e., everywhere divergent, power series solving even nonlinear meromorphic systems of ODE can be interpreted as *asymptotic expansions* of certain solutions of the same system. This by now classical theory has been presented in many books on differential equations in the complex plane or related topics. The presentations I am most familiar with are the monographs of *Wasow* [281] and *Sibuya* [251]. The most important result in this context is, in Wasow's terminology, the *Main Asymptotic Existence Theorem*: It states that to every formal solution of a system of meromorphic differential equations, and every sector in the complex plane of sufficiently small opening, one can find a solution of the system having the formal one as its asymptotic expansion. This solution, in general, is not uniquely determined, and the proofs given for this theorem, in various degrees of generality, do not provide a truly efficient way to compute such a solution, say, in terms of the formal solution. In view of these deficiencies of the Main Asymptotic Existence Theorem, the following two questions have been investigated in recent years:

1. Among all solutions of the linear system of ODE that are asymptotic to a given formal solution in some sector, can we characterize one uniquely by requiring certain additional properties?
2. If so, can we find a representation for this unique solution, in terms of the formal one, using an integral operator, or applying a summability method to the formal power series?

Parallel to these developments in the theory of ODE, scientists also founded a *general theory of asymptotic expansions*. Here, the analogue to the Main Asymptotic Existence Theorem is usually called *Ritt's theorem*, and is much easier to prove: Given any formal power series and any sector of arbitrary, but finite, opening, there exists a function that is holomorphic in this sector and has the formal power series as its asymptotic expansion. This function is never uniquely determined – not even when the power series converges, because it may not converge toward this function! To overcome this nonuniqueness, *Watson* [282, 283] and *Nevanlinna* [201] introduced a special kind of asymptotic expansions, now commonly called *of Gevrey order* $s > 0$. These have the property that the analogue to Ritt's theorem holds for sectors of opening up to $s\pi$, in which cases the function again is not uniquely determined. If the opening is larger than $s\pi$, however, a function that has a given formal power series as expansion of Gevrey order $s > 0$ may not exist, *but if it does, then it is uniquely determined.* In case of existence, the function can be represented as Laplace transform of another function that is holomorphic at the origin, and whose power series expansion is explicitly given in terms of the formal power series.

This achievement in the general theory of asymptotic expansions obviously escaped the attention of specialists for differential equations in the complex domain for quite some time: In a series of papers, *Horn* [125–128] showed the following result for linear systems of ODE: Let the leading term of the coefficient matrix, at a singularity of second kind, have all distinct eigenvalues. Then for sectors with sufficiently large opening one has uniqueness in the Main Asymptotic Existence Theorem, and the solution can be represented as a Laplace integral, or equivalently, in terms of (inverse) factorial series. However, he did not relate his observations to the general results of Watson and Nevanlinna. Later, *Trjitzinsky* [266] and *Turrittin* [268] treated somewhat more general situations, and they also pointed out the limitation of this approach to special cases.

In 1978/80, *Ramis* [225, 226] introduced his notion of k-summability of formal power series, which may best be interpreted as a formalization of the ideas of Watson and Nevanlinna. In this terminology, we may interpret the results of Horn, Trjitzinsky, and Turrittin as saying that, under additional assumptions, formal solutions of linear meromorphic systems of ODE are k-summable, for a suitable value of $k > 0$. In the general case, Ramis proved that every formal fundamental solution of every such system can be factored into a finite product of matrix power series (times some explicit functions), so that each factor is k-summable, but with the number k depending upon the factor. In my treatment of *first-level formal solutions* [8–10, 12, 40] I had, more or less by accident, independently obtained the same result. However, this factorization of formal solutions is not truly effective, and the factors are not uniquely determined, so that this result does not really allow us to compute the resulting matrix function from the formal series.

More recently, *Ecalle* [94–96] presented a way to achieve this computation, introducing his definition of *multisummability.* In a sense, his method differs from Ramis's definition of k-summability by cleverly enlarging the class of functions to which the Laplace operator, in some generalized form, can be applied. Another way of interpreting his method is by realizing that Ramis's k-summability in essence is a special case of what is called a *Moment Summability Method.* The rapidly growing coefficients of the formal power series are divided by a *moment sequence* to produce a convergent power series; then, the resulting holomorphic function is inserted into an integral operator, and the function so obtained is regarded as the sum of the original power series. Multisummability may be seen as the concept of iterating certain Moment Summability Methods. Given a power series with a radius of convergence equal to zero, we divide by a moment sequence as in the case of k-summability. The resulting series, however, will in general not be convergent, but instead is summable by some other Moment Method, which then produces a function to which the integral operator can be applied.

The idea of iteration of summability methods was already familiar to *Hardy* [112] and *Good* [110], but it was Ecalle who realized its potential with respect to formal solutions of ODE. He stated without proofs a large number of results concerning properties and applications of multisummability, e.g., to formal solutions of nonlinear systems of ODE. Based upon the described factorization of formal solutions in the linear case, it was more or less evident that multisummability applied to all formal solutions of linear systems. However, for nonlinear ones, the first complete proof for this fact was given by *Braaksma* [70]. Independent proofs, using different techniques, are due to *Ramis and Sibuya* [230] and myself [21].

Before Ecalle founded the theory of multisummability, researchers attacked the first one of the two questions raised above following different approaches: The main reason for wanting to have a unique solution that, say, near the point infinity, is asymptotic to the formal fundamental solution in some (small) sector lies in the fact that one wanted to have a clear description of what is called *Stokes' phenomenon.* This term refers to the fact that in different sectors one finds different solutions with the same asymptotic behavior as the variable tends to infinity. One way out of the dilemma was to just *select* a solution for each sector, study their interrelations, i.e., their dependence upon the sector, and then investigate how these relations change when one picks a different set of solutions. This line of approach leads to a description of Stokes' phenomenon in terms of equivalence classes of matrices; see *Sibuya*'s monograph [251] and the literature quoted there.

A somewhat more explicit approach was investigated by *Jurkat, Lutz*, and myself in [33, 34, 36, 143]: We proved the existence and uniqueness of a family of *normal solutions* that in certain sectors are asymptotic to a formal fundamental solution and that have a Stokes' phenomenon of an especially simple form. Meanwhile, a different characterization of normal solutions as solutions of certain integral equations has also been obtained [23].

Both of the approaches mentioned above did not lead to a direct representation formula for the corresponding solutions. The theory of multisummability, however, provides such a representation for the normal solutions [31]. In addition, it also allows a detailed analysis of the Stokes phenomenon, if not to say the computation of the Stokes matrices, through what Ecalle has called *analysis of resurgent functions.* This general theory is very useful for many problems other than the computation of Stokes' matrices. Aside from Ecalle's work, an introduction to this area, along with some of its applications, can be found in a book by *Candelbergher, Nosmas, and Pham* [78].

For further presentations of the recent history of linear systems in the complex plane, partially taking a more geometrical or algebraical point of view, see *Bertrand* [51], *Majima* [181], *Babbitt and Varadarajan* [5], or *Varadarajan* [274]. Some historical remarks on asymptotic power series and summability can be found in *Ramis's* booklet [229].

Appendix A
Matrices and Vector Spaces

We assume the reader to be familiar with the basic theory of matrices and vectors with complex entries, but we shall outline some notation and state some additional results on matrix algebra resp. systems of linear equations used in the book.

Matrices, resp. vectors, will normally be denoted by upper case, resp. lower case, letters, while their entries will be corresponding lower case letters bearing two indices, resp. one. We expect the reader to know about the concepts of *characteristic polynomials, eigenvalues and eigenvectors of square matrices, triangular resp. diagonal matrices*, and we shall write $\operatorname{diag}[\lambda_1, \ldots \lambda_\nu]$ for the diagonal matrix with diagonal entries λ_μ. We denote by I, or I_ν, the *unit matrix* of type $\nu \times \nu$, and write e_k for its kth column, i.e., the kth unit vector. The symbol A^T shall indicate the transpose of a matrix A. The *spectrum* of A will be the set of all eigenvalues of A.

As usual, we say that two square matrices A and B are *similar* provided that an invertible matrix T exists for which $AT = TB$. A well-known important result from matrix theory states that every square matrix A is similar to one in *Jordan canonical form*, or to a *Jordan matrix*. Here, Jordan matrices are assumed to be *lower triangular* matrices. Their diagonal entries are equal to the eigenvalues, with equal ones appearing consecutively. Nonzero entries off the diagonal are always equal to one and occur only in such places in the first subdiagonal where the two corresponding eigenvalues coincide. In other words, a Jordan matrix is the *direct sum*, in the sense of Section A.2, of so-called *Jordan blocks*; such a block has the form $J = \lambda I + N$, $\lambda \in \mathbb{C}$, where N is the nilpotent matrix with ones in *all* places of the first subdiagonal, and zeros elsewhere. However, observe that the letters J and N are not only used for such Jordan blocks, but for general Jordan resp. nilpotent matrices as well! We emphasize that every Jordan matrix J can be written as $J = \Lambda + N$, with a diagonal matrix Λ and a nilpotent N, *so that both commute.* For nilpotent matrices N, the smallest natural number m with $N^m = 0$ will be called *the order of nilpotency of* N. This order equals the size of the largest Jordan block of N.

In estimates we use a fixed, but arbitrary, norm for matrices and vectors, which for both will be denoted by the same symbol $\| \cdot \|$, and which is assumed to obey the following additional rules:

$$\|I\| = 1, \quad \|A\,B\| \le \|A\|\,\|B\|, \quad \|A\,x\| \le \|A\|\,\|x\|,$$

whenever the products give sense. *One can take* $\|A\| = \sup_k \sum_j |a_{kj}|$, $\|x\| = \sup_k |x_k|$, but nowhere in the book will we use any concrete choice.

A.1 Matrix Equations

In dealing with systems of ODE in the complex domain, one frequently is required to solve matrix equations of the following form:

- Suppose that $A \in \mathbb{C}^{k \times k}$, $B \in \mathbb{C}^{j \times j}$, and $C \in \mathbb{C}^{k \times j}$, for $k, j \in \mathbb{N}$, are given. Find some, or the set of all, $X \in \mathbb{C}^{k \times j}$ such that

$$A\,X - X\,B = C. \tag{A.1}$$

Note that (A.1) is nothing but a system of linear equations for the entries of X, since one can form a vector $x \in \mathbb{C}^{j \cdot k}$, containing the entries of X in any order, and rewrite (A.1) in the form $\tilde{A}\,x = c$ with a quadratic matrix $\tilde{A}$ built from A and B, and the vector c containing the entries of C. We shall, however, not do this here, but instead work with (A.1) directly. The theory of such equations is well understood, and we here require two well-known lemmas ([281] et al.):

Lemma 24 *Suppose that $A \in \mathbb{C}^{k \times k}$ and $B \in \mathbb{C}^{j \times j}$, for $k, j \in \mathbb{N}$, have disjoint spectra, i.e., do not have an eigenvalue in common. Then for every $C \in \mathbb{C}^{k \times j}$ the matrix equation* (A.1) *has a unique solution $X \in \mathbb{C}^{k \times j}$.*

Proof: Let T be such that $J = T^{-1}\,B\,T$ is in (lower triangular) Jordan form. Then (A.1) holds if and only if $A\,\tilde{X} - \tilde{X}\,J = \tilde{C}$ with $\tilde{X} = X\,T$, $\tilde{C} = C\,T$. Hence we may restrict to the case of $B = J = \operatorname{diag}[\lambda_1, \ldots, \lambda_j] + N$, with the elements of the nilpotent matrix N equal to zero except for some ones directly below the diagonal. Denoting the columns of X resp. C by x_μ resp. c_μ, we see that (A.1) is equivalent to

$$(A - \lambda_\mu I)\,x_\mu = c_\mu + \delta_\mu\,x_{\mu+1}, \quad 1 \le \mu \le j,$$

with δ_μ either zero or one, and in particular $\delta_j = 0$. According to our assumption $A - \lambda_\mu I$ always is invertible; hence these equations allow us to compute the columns x_μ in reverse order. □

The situation where A and B have eigenvalues in common is slightly more complicated: There may be no solution, and if there is one, it will not be unique. Here, we only need the following special case:

Lemma 25 *Let $J_1 \in \mathbb{C}^{k\times k}$ and $J_2 \in \mathbb{C}^{j\times j}$ be two Jordan blocks having the same eigenvalue, and assume $k \geq j$ (resp. $k \leq j$). Then for every $C \in \mathbb{C}^{k\times j}$ there exists a unique matrix $B \in \mathbb{C}^{k\times j}$ having nonzero entries in the first row (resp. last column) only, such that the matrix equation*

$$J_1 X - X J_2 = C - B \tag{A.2}$$

has a solution $X \in \mathbb{C}^{k\times j}$, which is unique within the set of matrices X having zero entries in the last row (resp. first column).

Proof: First, observe that we may restrict ourselves to the case where both matrices J_k are nilpotent. In the first case of $k \geq j$, denoting the entries in the first row of B by $\beta_1, \ldots, \beta_j$, and the columns of X resp. C by x_m resp. c_m, equation (A.2) is equivalent to

$$J_1 x_j = c_j - \beta_j e_1, \quad J_1 x_m = x_{m+1} + c_m - \beta_m e_1, \quad 1 \leq m \leq j-1,$$

with e_1 being the first unit vector. Computing $J_1 x_j$, one finds that the first equation is solvable if and only if β_j equals the first entry in c_j. If so, the entries in x_j can be uniquely computed except for the last one which remains undetermined. Inserting into the next equation with $m = j-1$, assuming $j \geq 2$, we again conclude that this equation is solvable if and only if β_{j-1} is chosen such that the first entry in the right hand side vector equals zero. Also note that $k \geq j$ implies that the undetermined entry in x_j does not interfere with the determination of β_{j-1}. Solving for x_{j-1} then gives a vector whose components are uniquely determined except for the last two, since the undetermined entry in x_j enters into the second last component of x_{j-1}. Repeating these arguments for the remaining columns then completes the proof of the first case. In the second case of $k \leq j$ one proceeds analogously, working with rows instead of columns. □

Note that both lemmas remain correct for $j = 1$ and/or $k = 1$; in particular the second one holds trivially for $j = k = 1$ with $B = C$, $X = 0$.

Exercises:

1. Solve (A.1) for

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. Spell out Lemma 25 for the cases of $j = 1$ and $k \geq 2$, resp. $j \geq 2$ and $k = 1$, resp. $j = k = 1$.

3. Suppose that (A.1) had been rewritten as $\tilde{A}\, x = c$, $x, c \in \mathbb{C}^{\,j \cdot k}$. Determine the eigenvalues of $\tilde{A}$ in terms of those of A and B.

A.2 Blocked Matrices

Let a matrix $A \in \mathbb{C}^{\nu \times \nu}$, $\nu \in \mathbb{N}$, be given.[1] We shall frequently have reason to *block* A into submatrices

$$A = [A_{jk}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1\mu} \\ A_{21} & A_{22} & \dots & A_{2\mu} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\mu 1} & A_{\mu 2} & \dots & A_{\mu\mu} \end{bmatrix},$$

where A_{jk} is of size $\nu_j \times \nu_k$, with $\nu_1 + \ldots + \nu_\mu = \nu$. So, in particular, all diagonal blocks A_{jj} are square matrices. If the block sizes are not obvious from the context, we shall say that A is blocked *of type* $(\nu_1, \ldots, \nu_\mu)$. If several matrices are blocked at a time, then it shall go without saying that all are blocked of the same type. If $A = [A_{jk}]$, $B = [B_{jk}]$ are blocked, then $C = AB = [C_{jk}]$ is also blocked, with $C_{jk} = \sum_{\nu=1}^{\mu} A_{j\nu} B_{\nu k}$. So matrix multiplication *respects the block structure*, and the same holds trivially for addition. We shall say that a matrix is *upper- resp. lower-triangularly blocked* with respect to a given type, if all blocks below resp. above the block-diagonal vanish, and accordingly we speak of *diagonally blocked matrices*. We use the symbol $\operatorname{diag}[A_1, \ldots, A_\mu]$ for the diagonally blocked matrix A with diagonal blocks A_k. This is the same as saying that A is the *direct sum* of the matrices A_k. For a triangularly blocked matrix A, the spectrum of A is the union of the spectra of the diagonal blocks A_{jj}, $1 \leq j \leq \mu$, and the inverse of A, in case it exists, is likewise triangularly blocked.

Exercises:

1. For

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

show that A is invertible if and only if both diagonal blocks are invertible, and then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1} A_{21} A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}.$$

[1] Note that we assume the set of natural numbers $\mathbb{N}$ *not to include zero.*

2. For
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
with A_{11} invertible, show
$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}\, A_{11}^{-1}\, A_{12} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix}.$$

3. For A as above, conclude $\det A = \det A_{11}\ \det(A_{22} - A_{21} A_{11}^{-1} A_{12})$ and compute A^{-1}, in case it exists.

A.3 Some Functional Analysis

In several chapters of the book we study functions with values in a Banach space. In this context, we shall make use of some standard results of Functional Analysis, in particular, of *Hahn-Banach's theorem* and the basic theory of continuous linear operators. While for the elementary theory of Banach spaces we refer to standard texts, we shall briefly outline some notation which we shall use:

- Given a vector space $\mathbb{E}$ over the field of complex numbers, assume that an operation $\cdot : \mathbb{E} \times \mathbb{E} \longmapsto \mathbb{E}$ is defined. We say that $\mathbb{E}$ is *an algebra* over $\mathbb{C}$, if for elements $a, b, c \in \mathbb{E}$ and $\alpha \in \mathbb{C}$ the two *associative laws*
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \alpha(a \cdot b) = (\alpha a) \cdot b$$
and the two *distributive laws*
$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$
always hold. If the commutative law $a \cdot b = b \cdot a$ also holds, then we say that $\mathbb{E}$ is a *commutative algebra.*

  The operation $\cdot$ will be referred to as *the multiplication* in $\mathbb{E}$ and for convenience we shall write $a\, b$ instead of $a \cdot b$.

- Let $\mathbb{E}$ be an algebra over $\mathbb{C}$. If an element $e \in \mathbb{E}$ exists so that for every $a \in \mathbb{E}$ we have $e\, a = a\, e = a$, then e is called *unit element* in $\mathbb{E}$. Observe that $\mathbb{E}$ can contain at most one such e, since if $\tilde{e}$ also is a unit element, then we have $\tilde{e} = e\, \tilde{e} = \tilde{e}\, e = e$.

- Let $\mathbb{E}$ be an algebra over $\mathbb{C}$ with unit element e. An element $a \in \mathbb{E}$ is called *invertible*, if some $b \in \mathbb{E}$ exists with $a\, b = b\, a = e$, and b then is called *inverse of* a. Again, it follows that only one such b can exist:

If $\tilde{b}$ also is an inverse of a, then $\tilde{b} = \tilde{b}\, e = \tilde{b}\, a\, b = e\, b = b$ follows, using the associative law. It is common to write a^{-1} for the inverse of a, in case it exists.

- Let $\mathbb{E}$ be an algebra over $\mathbb{C}$. A linear mapping $d:\ \mathbb{E} \times \mathbb{E} \longmapsto \mathbb{E}$ is called *a derivation* provided that for any $a, b \in \mathbb{E}$ the *product rule*

$$d(a\, b) = d(a)\, b + a\, d(b)$$

holds. If such a derivation is defined, then $\mathbb{E}$ is called *a differential algebra.*

- Let $\mathbb{E}$ be a vector space over $\mathbb{C}$. A mapping $\|\cdot\|:\ \mathbb{E} \longmapsto \mathbb{R}$ is called *a norm on* $\mathbb{E}$ if for any $a, b \in \mathbb{E}$ and $\alpha \in \mathbb{C}$ we have

  (a) $\|a\| \geq 0, \quad \|a\| = 0 \iff a = 0.$
  (b) $\|\alpha a\| = |\alpha|\, \|a\|.$
  (c) $\|a + b\| \leq \|a\| + \|b\|.$

Given such a norm, one can define *convergent sequences*, resp. *Cauchy sequences* in $\mathbb{E}$, as for sequences of real or complex numbers, by replacing the absolute value sign by the notion of norm. Doing so, we say that $\mathbb{E}$ is a *Banach space*, if every Cauchy sequence converges.

- Let $\mathbb{E}$ be an algebra on which a norm $\|\cdot\|$ is defined. If additionally to (a), (b), (c) we have

$$\|ab\| \leq \|a\|\, \|b\|,$$

then $\mathbb{E}$ is called *a normed algebra.* If $\mathbb{E}$ even is a Banach space, we speak of a *Banach algebra.* Note that, in case $\mathbb{E}$ has a unit element e, $\|e\| \geq 1$ follows, but equality not necessarily holds. However, one can always replace the given norm on $\mathbb{E}$ by another one, differing from the former one by a constant factor, so that then we do have $\|e\| = 1$, *and we shall always assume this to be the case.*

Let V be a Banach space. An arbitrary mapping F, defined on a subset $D \subset X$, is said to have a *fixed point* if for some $v \in D$ we have $F(v) = v$. Several theorems show existence of such a fixed point; here we are going to use the most elementary one:

Theorem 64 (Banach's Fixed Point Theorem)
Let D be a closed subset of a Banach space V, and let $F:\ D \longrightarrow D$ be a contraction; i.e., for some $\alpha \in (0, 1)$ let

$$\|F(v_1) - F(v_2)\| \leq \alpha\, \|v_1 - v_2\|,$$

for any two $v_1, v_2 \in D$. Then F has a unique fixed point in D.

For a proof of this theorem we refer to *Maddox* [179], or other books on (functional) analysis. We also mention that the theorem holds true whenever D is a *complete metric space.* In Chapter 3 we shall use Banach's fixed point theorem in the space of functions holomorphic in a disc and continuous up to the boundary; this is a Banach space with the norm being the supremum of the modulus of the function. Here, we apply the theorem to systems of *Volterra integral equations* of a very special form, which play a role in Section 8.2:

Let S be a fixed sector of possibly infinite radius,[2] let $r \in \mathbb{N}$ be fixed, and define $E(x) = \sum_{n=1}^{\infty} x^{n-r}/\Gamma(n/r)$.[3] For $c > 0$, consider the set V_c of functions x, holomorphic in S and so that $\|x\|_c = \sup_S |x(u)|/E(c|u|) < \infty$. It is easy to verify that V_c is, in fact, a Banach space. Similarly, the sets V_c^{μ}, resp. $V_c^{\mu\times\mu}$, of μ-vectors, resp. $\mu \times \mu$-matrices, of such functions are again a Banach space, provided we define their norm as above, but with modulus of $x(u)$ replaced by the corresponding vector resp. matrix norm. It is easy to check that all three Banach spaces become larger once we increase c.

For $\mu \in \mathbb{N}$ and $k > 0$, fix some $K \in V_k^{\mu\times\mu}$ and $f(u) \in V_k^{\mu}$. Moreover, let $A(u)$ be a $\nu \times \nu$ matrix holomorphic in S, invertible and so that $a = \sup_S \|A^{-1}(u)\| < \infty$. Under these assumptions, consider the integral equation

$$A(u)\,x(u) = f(u) + \int_0^u K((u^r - t^r)^{1/r})\,x(t)\,dt^r. \tag{A.3}$$

Then the following holds:

Proposition 26 *Under the above assumptions, the above integral equation has a unique solution $x \in V_\kappa^{\mu}$, with $\kappa > k + a\,\|K\|_k\,k^{1-r}$.*

Proof: By termwise integration, and using the *Beta Integral* (p. 229), one can see

$$\int_0^1 E(k|u|(1-x)^{1/r})\;E(\kappa|u|x^{1/r})\;dx = (k|u|)^{-r}\sum_{n=2}^{\infty}\frac{(\kappa|u|)^{n-r}}{\Gamma(n/r)}\sum_{j=1}^{n-1}(k/\kappa)^j,$$

which is at most $k^{1-r}\,|u|^{-r}\,(\kappa - k)^{-1}\,E(\kappa|u|)$. By assumption we have $\|K(u)\| \le \|K\|_k\,E(k|u|)$ for every $u \in S$. With $\|x(u)\| \le \|x\|_\kappa\,E(\kappa|u|)$, one can use the above inequality to show

$$\left\|\int_0^u K((u^r - t^r)^{1/r})\,x(t)\,dt^r\right\| \le \frac{\|K\|_k\,\|x\|_\kappa\,E(\kappa|u|)}{k^{r-1}\,(\kappa - k)}, \quad z \in S.$$

[2]See p. 60 for this notion.

[3]Observe that the function E is slightly different from, but intimately related to, *Mittag-Leffler's function* $E_{1/r}$, defined on p. 233. For the definition of the Gamma function $\Gamma(z)$, see p. 227.

From this we can see that the mapping

$$x \longmapsto Tx, \quad (Tx)(u) = A^{-1}(u) \left[f(u) + \int_0^u K((u^r - t^r)^{1/r})\, x(t)\, dt^r \right]$$

is contractive on V_κ^μ. Thus, Banach's fixed point theorem completes the proof. $\square$

Besides the *linear* integral equation studied above, we will investigate some *nonlinear systems* in the proofs of Theorem 11 (p. 52) and Lemma 11 (p. 124). The estimates needed in the proof of the lemma are more subtle, however; so we cannot directly apply the arguments used here.

Exercises:

1. Let $f(z)$ be holomorphic in a region $G \subset \mathbb{C}$, and for $z_0 \in G$ assume $f(z_0) = 0$, $f'(z_0) \neq 0$. For sufficiently small $\delta > 0$, show that $\phi(z) = z - f(z)/f'(z)$ maps the disc about z_0 of radius δ into itself and is a contraction there.

2. Investigate ϕ (as above) for the case of $f(z_0) = \ldots = f^{(n-1)}(z_0) = 0$, $f^{(n)}(z_0) \neq 0$, $n \geq 2$.

3. Under the assumptions of Proposition 26, show that if the vector f remains bounded at the origin (in S), or even is continuous there, then so is the solution vector x.

Appendix B
Functions with Values in Banach Spaces

Here, we shall briefly provide some results from the theory of holomorphic functions with values in a Banach space which are used in Chapters 4–7, 10, and 11. Those who are not interested in such a general setting may concentrate on functions with values in $\mathbb{C}$.

Throughout the book, we shall consider fixed, but arbitrary, Banach spaces $\mathbb{E}$ and $\mathbb{F}$ over the complex number field. We write $\mathcal{L}(\mathbb{E}, \mathbb{F})$ for the space of all linear continuous operators from $\mathbb{E}$ into $\mathbb{F}$, which again is a Banach space, and even a Banach algebra in case $\mathbb{E} = \mathbb{F}$. Instead of $\mathcal{L}(\mathbb{E}, \mathbb{C})$ we write $\mathbb{E}^*$. If nothing else is said, we shall denote by G some fixed *region in the complex domain*, i.e., some nonempty open and connected subset of $\mathbb{C}$, and by f some mapping $f:\ G \longrightarrow \mathbb{E}$ (or $f:\ G \longrightarrow \mathcal{L}(\mathbb{E}, \mathbb{F})$). We then call f *holomorphic in* G if for every $z_0 \in G$ the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists.[1] We call f *weakly holomorphic in* G if for every continuous linear functional $\phi \in \mathbb{E}^*$ the ($\mathbb{C}$-valued) function $\phi \circ f$ is holomorphic in G. Clearly, holomorphy implies weak holomorphy, and we shall see that the converse holds true as well. This makes it not too much of a surprise that many results from the theory of functions of a complex variable immediately carry over to the Banach space situation.

[1] To be exact, the above quotient of differences should be interpreted as multiplication of the vector $f(z) - f(z_0)$ from the left by $(z - z_0)^{-1}$; we prefer the quotient notation in order to point out the close analogy to the scalar situation.

For an arbitrary function f, one can define the integral over f along a path, i.e., a rectifiable curve, γ in G to be the "limit" of the Riemann sums, in case it exists. As for the scalar case, one can prove its existence for every continuous f. Since holomorphy implies continuity, we can form integrals of holomorphic functions over closed paths in G and raise the question whether they always vanish. This we shall do in the next section.

B.1 Cauchy's Theorem and its Consequences

In what follows we will assume *Cauchy's theorem* for holomorphic functions with values in $\mathbb{C}$ as known, and from it derive the same for $\mathbb{E}$-valued functions. Moreover, we shall derive other results for $\mathbb{E}$-valued functions as well, which the reader may or may not know for $\mathbb{C}$-valued ones.

The following theorem is the key to all the results we shall obtain later:

Theorem 65 (Cauchy's Integral Theorem and Formula) *Assume that G is simply connected and $f : G \to \mathbb{E}$ is continuous. Then the following statements are equivalent:*

(a) *The function f is weakly holomorphic in G.*

(b) *The function f is holomorphic in G.*

(c) *The integral of f over any closed path in G vanishes.*

(d) *For every positively oriented Jordan path γ and every z in the interior region of γ we have*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}\, dw. \tag{B.1}$$

Proof: Suppose (a) holds. For every path γ in G and every $\phi \in \mathbb{E}^*$, one can conclude from the definition of the integral via Riemann sums and linearity of ϕ that

$$\int_\gamma \phi(f(z))\, dz = \phi(x), \quad x = \int_\gamma f(z)\, dz. \tag{B.2}$$

The integral on the left vanishes for a closed path γ. Hence we conclude that $x = \int_\gamma f(z)\, dz$ satisfies $\phi(x) = 0$ for every $\phi \in \mathbb{E}^*$. This implies $x = 0$, according to *Hahn-Banach's theorem*. Thus we see that (c) follows. Conversely, it follows from (B.2) that (c) implies (a). Denoting the right-hand side of (B.1) by $g(z)$, we conclude similarly that weak holomorphy implies $\phi(f(z) - g(z)) = 0$ for every z in the interior region and every $\phi \in \mathbb{E}^*$, implying (B.1). This formula then can be seen to imply holomorphy of f, because differentiation under the integral sign is justified. Thus, the proof is completed, because we observed earlier that holomorphy implies weak holomorphy. □

By $\boldsymbol{H}(G, \mathbb{E})$ we shall denote the set of all $\mathbb{E}$-valued functions that are holomorphic in G. It is obvious from the definition that $\boldsymbol{H}(G, \mathbb{E})$ again is a vector space over $\mathbb{C}$. Moreover, if $f \in \boldsymbol{H}(G, \mathbb{E})$ and $T \in \boldsymbol{H}(G, \mathcal{L}(\mathbb{E}, \mathbb{F}))$ (so that $z \mapsto T(z)\, f(z)$ is a mapping from G into the Banach space $\mathbb{F}$), then one can show easily that $T f \in \boldsymbol{H}(G, \mathbb{F})$ and

$$[T(z)\, f(z)]' = T'(z) f(z) + T(z)\, f'(z), \quad z \in G.$$

Similarly, for $f \in \boldsymbol{H}(G, \mathbb{E})$ and $\alpha \in \boldsymbol{H}(G, \mathbb{C})$, we conclude $\alpha f \in \boldsymbol{H}(G, \mathbb{E})$ and

$$[\alpha(z)\, f(z)]' = \alpha'(z) f(z) + \alpha(z)\, f'(z), \quad z \in G.$$

Using Cauchy's integral formula (B.1) we now prove

Theorem 66 *Every $f \in \boldsymbol{H}(G, \mathbb{E})$ is infinitely often differentiable, and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw, \quad n \geq 0, \tag{B.3}$$

for γ and z as in (B.1).

Proof: Observe that differentiation under the integral in (B.1) can be justified, giving the desired result. □

Exercises: In the following exercises, let $\mathbb{E}$, $\mathbb{F}$ be Banach spaces and G some region in $\mathbb{C}$.

1. For $f \in \boldsymbol{H}(G, \mathbb{E})$ and $\phi \in \mathbb{E}^*$, show
$$\frac{d}{dz} \phi(f(z)) = \phi(f'(z)), \quad z \in G.$$
2. For $f \in \boldsymbol{H}(G, \mathbb{E})$, $T \in \boldsymbol{H}(G, \mathcal{L}(\mathbb{E}, \mathbb{F}))$ and $\alpha \in \boldsymbol{H}(G, \mathbb{C})$, prove the above statements on $T f$ resp. αf.

B.2 Power Series

As in the scalar case, one can represent holomorphic functions by infinite power series, as we show now:

For $f_n \in \mathbb{E}$, $n \geq 0$, consider the power series $\sum_{n=0}^\infty f_n\, (z - z_0)^n$ (note that again we slightly abuse notation and place the scalar factor $(z - z_0)^n$ to the right of the vectors f_n). Define

$$1/\rho = \limsup_{n \to \infty} \| f_n \|^{1/n}, \tag{B.4}$$

following the usual convention of $1/0 = \infty$, and vice versa. For every $K > 1/\rho$ we then have $\|f_n\| \le K^n$, for every $n \ge n_0$. For every $k < 1/\rho$, however, we have $\|f_n\| \ge k^n$ infinitely often. This shows that, as in the scalar case, the vector valued power series converges absolutely in $D(z_0, \rho)$ and uniformly in every strictly smaller disc, while it diverges for $|z| > \rho$.

Expanding $(w-z)^{-1} = \sum_{n=0}^\infty (z-z_0)^n (w-z_0)^{-n-1}$, inserting into (B.1) and interchanging summation and integration, which is justified because of uniform convergence (provided that z_0 is in the interior region of γ and $|z - z_0| < \inf_{w\in\gamma} |w - z_0|$), we obtain as for the scalar situation:

Proposition 27 *For $z_0 \in G$, let $\rho > 0$ be so that $D(z_0, \rho) \subset G$. Then for every $f \in \boldsymbol{H}(G, \mathbb{E}\,)$ we have*

$$f(z) = \sum_{n=0}^{\infty} f_n\,(z - z_0)^n, \quad z \in D(z_0, \rho), \tag{B.5}$$

with coefficients given by

$$f_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{|z-z_0|=\rho-\varepsilon} \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n \ge 0,$$

for every sufficiently small $\varepsilon > 0$.

Convergent power series with common center point z_0 may be added termwise, but in general the product of two power series is undefined. However, the following holds true:

Proposition 28 *For $f \in \boldsymbol{H}(G, \mathbb{E}\,)$, $\alpha \in \boldsymbol{H}(G, \mathbb{C}\,)$ and $T \in \boldsymbol{H}(G, \mathcal{L}(\mathbb{E}\,, \mathbb{F}))$ assume $f(z) = \sum_0^\infty f_n\,(z - z_0)^n$, $\alpha(z) = \sum_0^\infty \alpha_n\,(z - z_0)^n$ and $T(z) = \sum_0^\infty T_n\,(z - z_0)^n$, for $|z - z_0| < \rho$. Then $T\,f \in \boldsymbol{H}(G, \mathbb{F})$, $\alpha f \in \boldsymbol{H}(G, \mathbb{E}\,)$, and we have*

$$T(z)\,f(z) = \sum_0^\infty \left(\sum_{m=0}^n T_{n-m} f_m \right) (z - z_0)^n, \quad |z - z_0| < \rho,$$

$$\alpha(z)\,f(z) = \sum_0^\infty \left(\sum_{m=0}^n \alpha_{n-m} f_m \right) (z - z_0)^n, \quad |z - z_0| < \rho.$$

The simple proof is left to the reader. Note that if $\mathbb{E}$ is a Banach algebra, i.e., a product between elements of $\boldsymbol{H}(G, \mathbb{E}\,)$ is defined, then every $f \in \boldsymbol{H}(G, \mathbb{E}\,)$ can be identified with an element of $\boldsymbol{H}(G, \mathcal{L}(\mathbb{E}\,, \mathbb{E}\,))$. Hence it follows from the above propositions that $\boldsymbol{H}(G, \mathbb{E}\,)$ is a differential algebra. In particular, this holds for $\mathbb{E} = \mathbb{C}$.

Let f be given by (B.5) and choose $z_1 \in D(z_0, \rho)$. Then the power series representation of f about the point z_1 can be seen to be given by

$$f(z) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \binom{n+m}{m} f_{n+m}\,(z_1 - z_0)^n \right) (z - z_1)^m,$$

and the series converges at least for $|z - z_1| < \rho - |z_1 - z_0|$ but may converge in a larger disc. This is important in our discussion of holomorphic continuation in the next section.

We close this section with a brief discussion of holomorphic functions defined by integrals. While the reader should be familiar with the concept of locally uniform convergence of series and the fact that a function represented by such a series is holomorphic once each term is so, it may be not entirely clear what we mean by locally uniform convergence of an integral: For $a < b$, where $a = -\infty$ and/or $b = +\infty$ may occur, let $f(t, z)$ be an $\mathbb{E}$-valued function of a real variable $t \in (a, b)$ and a complex variable $z \in G$, and define

$$g(z) = \int_a^b f(t, z)\, dt. \tag{B.6}$$

We then say that the above integral *converges absolutely and locally uniformly in G*, if for every $z_0 \in G$ we can find an $\varepsilon > 0$ and a function $b(t)$, depending upon z_0 and ε, but independent of z, such that $\|f(t, z)\| \le b(t)$ for $|z - z_0| \le \varepsilon$ and every $t \in (a, b)$, and so that $\int_a^b b(t)\, dt$ exists. Then we have the following result on holomorphy of g:

Lemma 26 *Suppose that $f(t, z)$ is continuous in $t \in (a, b)$, for fixed $z \in G$, and holomorphic in $z \in G$, for fixed $t \in (a, b)$, and the integral* (B.6) *converges absolutely and locally uniformly in G. Then $g(z)$ is holomorphic for z in G.*

Proof: Observe that $\|f(t, z)\| \le b(t)$ for $|z - z_0| \le \varepsilon$ implies (using Cauchy's formula) $f(t, z) = \sum_0^\infty f_n(t)\, (z - z_0)^n$, with $\|f_n(t)\| \le \varepsilon^{-n}\, b(t)$. Hence termwise integration of the series is justified for $|z - z_0| < \varepsilon$, and doing so gives the power series representation of $g(z)$ about the point z_0, thus proving its holomorphy. □

Exercises:

1. Let a power series $\sum_0^\infty f_n\, (z - z_0)^n$, with coefficients $f_n \in \mathbb{E}$, converge for $|z - z_0| < \rho$, $\rho > 0$, defining an analytic function f in $D = D(z_0, \rho)$. Assume $f(z_k) = 0$ for values $z_k \ne z_0$ in D with $\lim_{k\to\infty} z_k = z_0$. Show that then $f_n = 0$, $n \ge 0$, hence $f(z) \equiv 0$.

2. Let $f \in \boldsymbol{H}(R, \mathbb{E})$, $R = \{z : \ \rho_1 < |z - z_0| < \rho_2\}$.

   (a) For $\rho_1 < \rho < \rho_2$, define

   $$f^{\pm}(z) = \frac{1}{2\pi i} \oint_{|w - z_0| = \rho} \frac{f(w)}{w - z}\, dw, \quad \pm|z - z_0| > \pm\rho.$$

   Show that $f^{\pm}(z)$ is independent of ρ (provided that $\pm|z - z_0| > \pm\rho$ remains valid). Conclude that f^+, resp. f^-, is analytic for

$|z - z_0| > \rho_1$, resp. for $|z - z_0| < \rho_2$, and show

$$f(z) = f^-(z) - f^+(z), \quad z \in R.$$

(b) Show that f can be expanded into a *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} f_n \, (z - z_0)^n, \quad z \in R,$$

with coefficients $f_n \in \mathbb{E}$ given by

$$f_n = \frac{1}{2\pi i} \oint_{|w - z_0| = \rho} f(w) \, (w - z_0)^{-n-1} \, dw.$$

(c) In case of $\rho_1 = 0$, define the notions of *essential singularity, resp. pole, resp. removable singularity* as saying that, with f_n as defined above, we have $f_n \neq 0$ for infinitely many negative n, resp. $f_{n_0} \neq 0$ for some negative n_0 and $f_n = 0$ for $n < n_0$, resp. $f_n = 0$ for every negative n. Show that $f(z)$ being bounded for $0 < |z - z_0| \leq \rho < \rho_2$ implies that z_0 is a removable singularity of f.

3. Let G be a simply connected region, and let f be $\mathbb{E}$-valued and holomorphic in G, except for singularities at points $z_n \in G$, $n \geq 0$, which do not accumulate in G. Define the *residue* of f at a point z_n as the coefficient f_{-1} in the Laurent expansion of f about z_n. Show the *Residue Theorem*:

   The integral of f over any positively oriented Jordan curve in G which avoids the points z_n, $n \geq 0$, equals $2\pi i$ times the sum of the residues of f at those points z_n which are in the interior region of the curve.

4. If $G = G_1 \cup G_2$, for arbitrary regions G_k, and if $f_k \in \boldsymbol{H}(G_k, \mathbb{E})$ satisfy $f_1(z) = f_2(z)$ for every $z \in G_1 \cap G_2$, then show that there is precisely one $f \in \boldsymbol{H}(G, \mathbb{E})$ with $f(z) = f_k(z)$ for $z \in G_k$.

5. For $f \in \boldsymbol{H}(\mathbb{C}, \mathbb{E})$, i.e., an entire $\mathbb{E}$-valued function, show that if f is bounded, then f is necessarily constant.

B.3 Holomorphic Continuation

Let $f \in \boldsymbol{H}(G, \mathbb{E})$ and an arbitrary path γ with parameterization $z(t)$, $0 \leq t \leq 1$, beginning at some point $z_0 \in G$ but terminating at a point $z_1 \notin G$, be given. Then we say that f *can be holomorphically continued along* γ, provided that we can find a partitioning $0 = t_0 < t_1 < \ldots < t_m = 1$ and some $\varepsilon > 0$, such that $|z(t) - z(t_k)| < \varepsilon$ for $t_{k-1} \le t \le t_k$, $1 \le k \le m$, and so that f, by successive re-expansion of its power series about the points $z(t_k)$, can be defined in the discs $D(z(t_k), \varepsilon)$. Then, f obviously is holomorphic in every one of these discs, but perhaps not in their union, since when the path γ intersects with itself, f can in general not be unambiguously defined at intersection points.

Continuity of $z(t)$ implies existence of a number δ, $0 < \delta < \varepsilon$, so that $|z(t) - z(t_k)| \le \delta$ for $t_{k-1} \le t \le t_k$, $1 \le k \le m$. This may be used to show that f can also be analytically continued along every curve $\tilde{\gamma}$ with parameterization $\tilde{z}(t)$, $0 \le t \le 1$, having the same endpoints as γ, provided that $|z(t) - \tilde{z}(t)| \le (\varepsilon - \delta)/2$ for every t. In particular, *continuation along both curves leads to the same holomorphic function near* z_1. This is important in the proof of the following theorem:

Theorem 67 (Monodromy Theorem) *Let G be an arbitrary region, let $D = D(z_0, \rho) \subset G$, and let $f \in \boldsymbol{H}(D, \mathbb{E})$. Moreover, assume that f can be holomorphically continued along every path γ, beginning at z_0 and staying inside G. Then the following holds true:*

(a) *If γ_0, γ_1 are any two homotopic paths in G, both beginning at z_0 and ending, say, at z_1, then holomorphic continuation of f along either path produces the same value $f(z_1)$.*

(b) *If G is simply connected, then $f \in \boldsymbol{H}(G, \mathbb{E})$.*

Proof: Clearly, (b) follows from (a), since in a simply connected region any two paths with common endpoints are homotopic; hence by holomorphic continuation we may define $f(z)$ unambiguously at every point $z \in G$, and then f is holomorphic in G. To show (a), choose any two paths γ_0, γ_1 in G, beginning at z_0 and ending at a common point z_1. If the two paths are *homotopic*, then by definition there exists a continuous map $z(s, t)$ of the unit square $R = [0, 1] \times [0, 1]$ into G such that $z(0, \cdot)$, resp. $z(1, \cdot)$, is a parameterization of γ_0, resp. γ_1, while for arbitrary $s \in [0, 1]$ the function $z(s, \cdot)$ parameterizes some other path γ_s in G, connecting z_0 and z_1. By assumption f can be holomorphically continued along every one of these paths, and we denote by $f_s(z_1)$ the value of f at z_1, obtained by holomorphic continuation along γ_s. For every fixed s, we conclude from the definition of holomorphic continuation existence of $\varepsilon_s > 0$ such that

1. the path γ_s is covered by finitely many discs of radius ε_s,
2. each disc contains the center of the following one (when moving along the path),
3. f is holomorphic in each one of these discs.

Continuity of the function $z(s, t)$ on R implies existence of $\delta_s > 0$ so that for $\tilde{s} \in [0, 1]$ with $|\tilde{s} - s| < \delta_s$ the path $\gamma_{\tilde{s}}$ remains within the union of these discs, which implies $f_{\tilde{s}}(z_1) = f_s(z_1)$ for these values $\tilde{s}$, but this clearly implies that $f_s(z_1)$ has to be constant. □

While we have seen that functions can be holomorphically continued by successive re-expansion (provided that they *are* analytic at the points of some path), this method has little practical value. We shall, however, see in the following two subsections that other, more effective ways of holomorphic continuation are available in special cases. Moreover, both examples show that continuation along paths that are not homotopic may, or may not, produce the same value for the function at the common endpoint of the paths.

• The Natural Logarithm

The (natural) logarithm of a nonzero complex number z is best defined by the integral

$$\log z = \int_1^z \frac{dw}{w} = \log|z| + i \arg z. \tag{B.7}$$

According to the Monodromy Theorem, this definition is unambiguous in any simply connected region containing 1 but not the origin, since then it is irrelevant along which path integration is performed. However, since $\oint w^{-1} dw = 2\pi i$ when integrating along positively oriented circles about the origin, we see that *it is not possible to give an unambiguous definition of* $\log z$ *in the full punctured complex plane* (i.e., the complex plane with the origin deleted). Instead, one can think of the logarithm as defined in *the cut plane*, i.e., the complex plane with the negative real numbers including zero deleted, interpreting $\arg z$ to be in the open interval $(-\pi, \pi)$, or what is much better for our purposes, imagine $\log z$ as defined on what is called *the Riemann surface of the logarithm*: Think of a "spiraling staircase" with infinitely many levels in both directions, "centered" at the origin. Points on this staircase are uniquely characterized by their distance $r > 0$ from the origin and the angle φ, measured against some arbitrarily chosen direction. To every point on the staircase, or simply every pair (r, φ), there corresponds the complex number $z = r \exp[2\pi i]$, called *the projection.* Instead of a point, we can also consider the projection of subsets of the Riemann surface. Consider a curve in the *punctured complex plane*, i.e., the plane with the origin deleted, parameterized by $z(t)$, $0 \le t \le 1$. It can be "lifted" to the Riemann surface by choosing a value for $\arg z(0)$ and requiring that $\arg z(t)$ change continuously with respect to t. This implies that double points on the curve in the plane may correspond to distinct points on the surface. We shall use the same symbol z both for points in the punctured plane and on the Riemann surface, but we emphasize that on the Riemann surface the argument of a point is uniquely defined. For example, the points z and $z \exp[2\pi i]$, when considered on the Riemann surface, will not be the same; instead, the second one sits directly above the first one when using the visualization as a staircase. Taking this into account, the function $\log z$ can then be unambiguously defined by (B.7). It will be very convenient to view other functions $\sqrt{z}$, or z^α, having *branch points at the origin, as being defined on this Riemann surface*, in order to avoid discussing which one of their, in general infinitely many, branches one should take in a particular situation. For arbitrary functions f, defined on some domain G considered on the Riemann surface, we shall say that f *is single-valued* if $f(z) = f(z \exp[2\pi i])$ whenever both sides are defined (which may very well never happen). This is nothing else but saying that we may consider the function, instead of the domain G on the Riemann surface, as defined on the corresponding projection in the plane. For example, the function z^α is single-valued whenever $\alpha \in \mathbb{Z}$.

• The Gamma Function

The following $\mathbb{C}$-valued function plays a prominent role in this book as well as in many other branches of mathematics:

THE GAMMA FUNCTION

For z in the right half-plane, the function

$$\Gamma(z) = \int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,dt \tag{B.8}$$

(integrating along the positive real axis and defining the, in general multi-valued, function t^{z-1} according to the specification of $\arg t = 0$) is named *the Gamma function.*

It is easily seen that the integral in (B.8) converges absolutely and locally uniformly for z in the right half-plane; hence the function is holomorphic there, because the integrand is a holomorphic function of z. If, instead of starting from zero, we integrate from one to infinity, the corresponding function is entire, i.e., analytic in $\mathbb{C}$. In the remaining integral, we expand e^{-t} into its power series and integrate termwise to obtain

$$\Gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(z+n)} \qquad + \quad \int_1^\infty t^{z-1}\,\mathrm{e}^{-t}\,dt.$$

Clearly, the series in this representation formula remains convergent for z in the closed left half-plane except for the points $0, -1, \ldots$ where one of the terms becomes infinite; thus, this formula provides the holomorphic continuation of $\Gamma(z)$ into the complex plane except for first-order poles at the nonpositive integers. In particular, *holomorphic continuation along arbitrary paths avoiding these points is always possible and the value of the function at the endpoint is independent of the path*, because the function does not have any *branch points*. For completeness, and because these formulas will play a role in the book, we mention the following two different ways of obtaining the holomorphic continuation of $\Gamma(z)$:

- For z in the right half-plane, integrating (B.8) by parts implies

$$\Gamma(1+z) \;=\; z\,\Gamma(z). \tag{B.9}$$

  Solving for $\Gamma(z)$, we see that we obtain holomorphic continuation, first to values $z = x + i\,y$ with $x > -1$, except for a pole a the origin, then to those with $x > -2$, except for a pole at -1, etc. It is worthwhile to note that (B.9), together with $\Gamma(1) = 1$, implies $\Gamma(1+n) = n!$, $n \in \mathbb{N}$.

- One can show (compare the exercises below) the following integral representation for the reciprocal Gamma function, usually referred to as *Hankel's formula*:

$$\frac{1}{\Gamma(z)} \;=\; \frac{1}{2\pi i}\int_\gamma w^{-z}\,\mathrm{e}^{w}\,dw, \tag{B.10}$$

  where γ is the path of integration from infinity along the ray $\arg w = -\pi$ to the unit circle, then around the circle and back to infinity along the ray $\arg w = \pi$ (note that both rays are the negative real axis when pictured in the complex plane, but we better view them on the Riemann surface of the logarithm and define the branch of the, in general multivalued, function w^{-z} according to the specifications of $\arg w$). This integral can be shown to converge *for every* $z \in \mathbb{C}$, so the reciprocal Gamma function is holomorphic in the full plane. This again proves that $\Gamma(z)$ can be analytically continued into the left half-plane and has poles at all zeros of the reciprocal function – however, we do not directly learn the location of these poles, resp. zeros!

In this book, we shall frequently use the so-called *Beta Integral*: Define

$$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1}\,t^{\beta-1}\,dt, \quad \operatorname{Re}\alpha > 0, \;\; \operatorname{Re}\beta > 0.$$

To evaluate this integral, observe that by a simple change of variable we obtain

$$u^{\alpha+\beta-1}\,B(\alpha,\beta) = \int_0^u (u-t)^{\alpha-1}\,t^{\beta-1}\,dt, \qquad u \in \mathbb{C}.$$

On the other hand, the identity

$$\int_0^\infty \mathrm{e}^{-u}\int_0^u (u-t)^{\alpha-1}\,t^{\beta-1}\,dt\;du = \Gamma(\alpha)\,\Gamma(\beta)$$

follows by interchanging the order of integration and using the definition of the Gamma function. This, together with the integral for $\Gamma(\alpha+\beta)$, implies

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \operatorname{Re}\alpha > 0, \ \operatorname{Re}\beta > 0. \tag{B.11}$$

Finally, we show the following result upon the asymptotic behavior of the Gamma function for $z \to \infty$:

Theorem 68 (Stirling's Formula) *For $|z| \to \infty$ in sectors $|\arg z| \le \pi - \varepsilon$, for every $\varepsilon > 0$, we have*

$$\Gamma(z)\,\frac{e^z\sqrt{z}}{\sqrt{2\pi}\;z^z} \quad\to\quad 1. \tag{B.12}$$

More precisely, there is a formal power series $\hat f(z) = \sum_0^\infty f_n\,z^{-n}$, with constant term $f_0 = 1$, so that

$$\Gamma(z)\,\frac{e^z\sqrt{z}}{\sqrt{2\pi}\;z^z} \;\cong_1\; \hat f(z), \qquad |\arg z| < \pi,$$

in the sense of Section 4.5.

Proof: We give a proof especially suited to the theory developed in previous chapters; the reader may wish to verify that here we only use the definition of 1-summability and some results on integral equations, and in particular do not rely on anything for which we have used (B.12).

As was shown above, the Gamma function is a solution of the difference equation $x(z+1) = z\,x(z)$, and putting $y(z) = e^z\sqrt{z}\,z^{-z}x(z)$, we find $y(z+1) = a(z)\,y(z)$, with

$$a(z) = \frac{\exp[1 - z\log(1+1/z)]}{\sqrt{1+1/z}} \;=\; 1 + a_2\,z^{-2} + a_3\,z^{-3} + \ldots, \quad |z| > 1.$$

Formally setting $y(z) = \hat y(z) = \sum_0^\infty y_n\,z^{-n}$; hence

$$y(z+1) = \sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{n} \binom{-m}{n-m} y_m,$$

it is not difficult to see that, for arbitrary y_0, the power series formally satisfies the difference equation if and only if the remaining coefficients are given by the recursion

$$n\,y_n = \sum_{m=0}^{n-1}\left[\binom{-m}{n+1-m} - a_{n+1-m}\right] y_m, \qquad n \ge 1.$$

Set $t(u) = \sum_0^\infty u^n\,y_{n+1}/n!$, $k(u) = \sum_1^\infty u^n\,a_{n+1}/n!$, with a_n being the coefficients in the power series expansion of $a(z)$; in particular, observe

$a_1 = 0$. Then it is readily seen that formally $t(u)$ satisfies the integral equation

$$(\mathrm{e}^{-u} - 1)\, t(u) = y_0\, k(u) + \int_0^u k(u-w)\, t(w)\, dw.$$

Since $k(u)$ vanishes at the origin, we can divide this integral equation by u to eliminate the first-order zero of the factor $\mathrm{e}^{-u} - 1$ at the origin, and then arguments very similar to the ones used in the proof of Proposition 26 (p. 217) show the existence and uniqueness of a solution $t(u)$ of this integral equation that is analytic in the complex plane with cuts from $\pm 2\pi i$ along the positive resp. negative imaginary axes, because of the other zeros of $\mathrm{e}^{-u} - 1$, and the function $t(u)$ satisfies an estimate of the form

$$|t(u)| \le K\, \mathrm{e}^{c|u|}, \qquad |\arg u| \le \pi/2 - \varepsilon,$$

for every $\varepsilon > 0$ and sufficiently large $c, K > 0$ depending upon ε. The power series expansion of $t(u)$ then can be seen to be of the form as above; hence, by definition the formal series $\hat{y}(z)$ is 1-summable in all directions d with $|d| < \pi/2$, and its sum $y(z)$ satisfies the above difference equation, for every choice of y_0. Thus we are left to show that for $y_0 = \sqrt{2\pi}$ we have $y(z) = \mathrm{e}^z \sqrt{z}\, z^{-z} \Gamma(z)$, but this follows from Exercise 4. □

Exercises: In the following exercises, let G be a simply connected region.

1. Show that if $f \in \boldsymbol{H}(G, \mathbb{E})$ has a power series expansion about some $z_0 \in G$ in which all coefficients vanish, then f is identically zero.

2. Show that if $f \in \boldsymbol{H}(G, \mathbb{E})$ satisfies $f^{(n)}(z_0) = 0$ for some $z_0 \in G$ and every $n \ge 0$, then f vanishes identically. So in other words, if $f(z_0) = 0$ but $f(z) \not\equiv 0$, then for some $n \ge 1$ we must have $f^{(n)}(z_0) \ne 0$; the number z_0 then is called a *root* of f, and the minimal n with this property is called *the order of the root* of f.

3. Show the *Identity Theorem*: If $f \in \boldsymbol{H}(G, \mathbb{E})$ has infinitely many roots, accumulating at some $z_0 \in G$, then f is identically zero.

4. In this exercise, let x be a positive real number.

   (a) Show $\Gamma(x) = x^x \int_0^\infty \mathrm{e}^{-x(t - \log t)} \frac{dt}{t}$.

   (b) For $t > 0$, check that $u = t - 1 - \log t$ is strictly increasing and make a corresponding change of variable to show

   $$\int_1^\infty \mathrm{e}^{-x(t-\log t)} \frac{dt}{t} \;=\; \mathrm{e}^{-x} \int_0^\infty \mathrm{e}^{-xu} f_1(u)\, du,$$

   with a positive continuous function $f_1(u)$ satisfying $\sqrt{u}\, f_1(u) \to 1/\sqrt{2}$ as $u \to 0+$.

(c) For $0 < t < 1$, check that $u = t - 1 - \log t$ is strictly decreasing and make a corresponding change of variable to show

$$\int_0^1 \mathrm{e}^{-x(t-\log t)} \frac{dt}{t} = \mathrm{e}^{-x} \int_0^\infty \mathrm{e}^{-xu} f_2(u)\, du,$$

with a positive continuous function $f_2(u)$ satisfying $\sqrt{u}\, f_2(u) \to 1/\sqrt{2}$ as $u \to 0+$.

(d) Use the identities derived above to show

$$\Gamma(x) \frac{\mathrm{e}^x \sqrt{x}}{x^x} \to \sqrt{2\pi},$$

for (positive real) $x \to \infty$, and from this complete the proof of *Stirling's formula.*

5. For arbitrary $a \in \mathbb{C}$, use (B.12) to show for $|z| \to \infty$ in sectors $|\arg z| \le \pi - \varepsilon$, for every $\varepsilon > 0$,

$$\frac{\Gamma(z)\, z^a}{\Gamma(a+z)} \to 1. \tag{B.13}$$

6. Show

$$\frac{n^z\, n!}{(z+1)\cdot \ldots \cdot (z+n)} \to \Gamma(1+z), \quad n \to \infty.$$

7. In this and the following exercises we aim at proving (B.10). For this purpose, let the integral on the right of (B.10) be denoted by $M(z)$. Show that $M(z)$ is an entire function, having real values for $z \in \mathbb{R}$.

8. For $\lambda \in \mathbb{C}$ and arbitrary real τ, show

$$\frac{-1}{2\pi i} \int_{\gamma(\tau)} \mathrm{e}^{u/z} z^\lambda \frac{dz}{z} = M(\lambda)\, u^\lambda,$$

with a path of integration $\gamma(\tau)$ as follows: From the origin along $\arg z = \tau + (\varepsilon + \pi)/2$ to some z_1 of modulus r, then along the circle $|z| = r$ to the ray $\arg z = \tau - (\varepsilon + \pi)/2$, and back to the origin along this ray, with arbitrary $r > 0$ and $0 < \varepsilon < \pi/2$.

9. For $\operatorname{Re} \lambda > 0$ and arbitrary real τ, show

$$z^{-1} \int_0^{\infty(\tau)} u^\lambda \mathrm{e}^{-u/z} du = \Gamma(1+\lambda)\, z^\lambda,$$

with integration along the ray $\arg u = \tau$.

10. For Re $\lambda > 0$ and arbitrary real τ, show that

$$z^{-1} \int_0^{\infty(\tau)} \left[\frac{-1}{2\pi i} \int_{\gamma(\tau)} e^{u/z} z^{\lambda} \frac{dz}{z} \right] e^{-u/w} du = w^{\lambda},$$

and use this and the previous two exercises to conclude (B.10). Compare this to Section 5.5, where an analogous formula is obtained for a more general situation.

11. For Re $z < -1$, show that the path of integration γ in (B.10) can be replaced with a path from infinity to the origin along the ray $\arg w = -\pi$, and back to infinity along the ray $\arg w = \pi$, interpreting $w^{-z} = e^{-z \log w}$ accordingly. Use this to show

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad z \notin \mathbb{Z}. \tag{B.14}$$

In particular, conclude that $\Gamma(1/2) = \sqrt{\pi}$.

12. Use (B.14) and Exercise 6 to show *Wallis's* product formula:

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} (1 - z^2/k^2), \qquad z \in \mathbb{C}.$$

B.4 Order and Type of Holomorphic Functions

Let $\bar{S} = \{z : \; |z| \geq \rho_0, \; \alpha \leq \arg z \leq \beta\}$ and assume f holomorphic in the interior of $\bar{S}$ and continuous up to the boundary. Define $M(\rho, f) = \sup\{\|f(z)\| : \; \alpha \leq \arg z \leq \beta, |z| = \rho\}$, for $\rho \geq \rho_0$, and call

$$k = k(f) = \limsup_{\rho \to \infty} \frac{\log(\log M(\rho, f))}{\log \rho} \tag{B.15}$$

the exponential order (or for short: the order) of f (in $\bar{S}$). By definition we have $0 \leq k \leq \infty$. It is easily seen that one can also define k as the infimum of all $\kappa > 0$ for which a constant K exists such that

$$\|f(z)\| \leq K \exp[|z|^{\kappa}], \qquad z \in \bar{S}, \tag{B.16}$$

following the convention that the infimum of the empty set is ∞. If k is neither zero nor infinite, we refer to

$$\tau = \tau(f) = \limsup_{\rho \to \infty} \frac{\log M(\rho, f)}{\rho^k} \tag{B.17}$$

as *the type* of f and say that f is *of finite type* if $\tau < \infty$. As for the order, one can also define τ as the infimum of all $c > 0$ for which K exists such that

$$\|f(z)\| \le K \exp[c|z|^k], \quad z \in \bar{S}. \tag{B.18}$$

If $S = \{z : \ |z| > \rho_0, \ \alpha < \arg z < \beta\}$ and $f \in \boldsymbol{H}(S, \mathbb{E})$, we define its order as the supremum of all orders of f in closed subsectors of S, and analogously for the type. In particular, f is either of order smaller than k, or of order k and finite type, if an estimate (B.18) holds in every closed subsector of S, with constants c, K that may go off to infinity when we approach the boundary of S.

As is standard for $\mathbb{C}$-valued functions, we say that a function $f \in \boldsymbol{H}(\mathbb{C}, \mathbb{E})$ is an *entire function*. For such an entire f we take $M(\rho, f)$ as the maximum modulus of $f(z)$ on the circle of radius ρ (about the origin) and define order and type as above. As an example, we consider

Mittag-Leffler's Function

The ($\mathbb{C}$-valued) function

$$E_\alpha(z) = \sum_{n=0}^{\infty} z^n / \Gamma(1 + \alpha n), \quad \alpha > 0,$$

is called *Mittag-Leffler's function* of index α.

Using (B.10), one can show

$$E_\alpha(z) = \frac{1}{2\pi i} \int_\gamma \mathrm{e}^w \frac{w^{\alpha-1}}{w^\alpha - z} \, dw, \tag{B.19}$$

first for $|z| < 1$ and then, using Cauchy's theorem to change the path of integration, one can show a corresponding representation for arbitrary values of z. It is shown in the exercises below that E_α is an entire function of exponential order $k = 1/\alpha$ and type $\tau = 1$.

As for the classical case of $\mathbb{E} = \mathbb{C}$, order and type of entire functions can be expressed in terms of their power series coefficients:

Theorem 69 *Let $f \in \boldsymbol{H}(\mathbb{C}, \mathbb{E})$ have the power series expansion $f(z) = \sum f_n z^n$, $z \in \mathbb{C}$. Then the order k and, in case of $0 < k < \infty$, the type τ of f are given by*

$$k = \limsup_{n \to \infty} \frac{n \log n}{\log(1/\|f_n\|)}, \tag{B.20}$$

$$\tau = \frac{1}{\mathrm{e}k} \limsup_{n \to \infty} n \|f_n\|^{k/n}. \tag{B.21}$$

Proof: Assume $\|f(z)\| \le C \exp[c|z|^\kappa]$, for some $C, c, \kappa \ge 0$. Then we conclude from Proposition 27 (p. 222), with $z_0 = 0$ and arbitrary $\rho > 0$, that

$\|f_n\| \le C\,\rho^{-n}\,\exp[c\rho^\kappa]$, $n \ge 1$. The right-hand side, as a function of ρ, becomes minimal for $c\rho^\kappa = n/\kappa$; hence $\|f_n\| \le C\,(c\kappa/n)^{n/\kappa}\,\exp[n/\kappa]$, $n \ge 1$ follows. Conversely, such an estimate for f_n implies, using Stirling's formula, that $\|f_n\| \le \tilde C\,(Kc^{1/\kappa})^n/\Gamma(1+n/\kappa)$, $n \ge 1$, for every $K > 1$ and $\tilde C$ sufficiently large. Hence, according to the above discussion of Mittag-Leffler's function, we obtain $\|f(z)\| \le \tilde C\,E_{1/\kappa}(Kc^{1/\kappa}|z|) \le \tilde C\,\mathrm{e}^{\tilde c|z|^\kappa}$, for every $\tilde c > c$. Using this, one can easily complete the proof. □

Exercises: In the following exercises, let $\alpha > 0$ and consider Mittag-Leffler's function $E_\alpha(z)$.

1. Assume $\alpha < 2$, and let $\bar S = \{z : \ |\arg z| \le \alpha\pi/2$. Use (B.19) and Cauchy's integral theorem to show for $s \in \bar S$ and $|z| > 1$ that
$$E_\alpha(z) \;=\; \alpha^{-1}\mathrm{e}^{z^{1/\alpha}} \;+\; \frac{1}{2\pi i}\int_\gamma \mathrm{e}^w\,\frac{w^{\alpha-1}}{w^\alpha - z}\,dw,$$
and the integral tends to zero as $z \to \infty$.

2. Let α and $\bar S$ be as in the previous exercise. Use (B.19) and Cauchy's integral theorem to show that $E_\alpha(z)$ remains bounded in arbitrary closed subsectors that do not intersect with $\bar S$. Use this and the previous exercise to show that $E_\alpha(z)$ is of order $k = 1/\alpha$ and type $\tau = 1$.

3. Compute order and type of $E_\alpha(z)$, for arbitrary $\alpha > 0$.

B.5 The Phragmén-Lindelöf Principle

In the literature one finds a number of theorems, all bearing the name *Phragmén-Lindelöf* and being variants, resp. consequences, of the well-known *Maximum Modulus Principle*, which easily generalizes to Banach space valued functions:

Proposition 29 (Maximum Modulus Principle) *Let G be any region in $\mathbb{C}$, and assume that some z_0 and $\rho > 0$ with $D(z_0,\rho) \subset G$ exist for which we have*
$$\|f(z)\| \le \|f(z_0)\|, \quad z \in D(z_0,\rho).$$
Then $f(z)$ is constant.

Proof: Without loss of generality, let $z_0 = 0$. Expanding $f(z) = \sum_0^\infty f_n\,z^n$, $|z| < \rho$, we wish to show $f_n = 0$ for $n \ge 1$. Suppose we did so for $1 \le n \le m-1$, which is an empty assumption for $m = 1$. According to *Hahn-Banach's theorem*, there exists $\phi \in \mathbb{E}^*$ with $\|\phi\| = 1$ and

$\phi(f_0) = \|f_0\|$, $|\phi(f_m)| = \|f_m\|$. By continuity of ϕ we conclude $\phi(f(z)) = \phi(f_0) + \sum_{n=m}^{\infty} \phi(f_n)\, z^n$, $|z| < \rho$. The assumptions on f and ϕ imply $|\phi(f(z))| \le \|f(z)\| \le \|f(0)\| = \phi(f_0)$. Taking z so that $z^m\, \phi(f_m)$ and $\phi(f_0)$ have the same argument, and $|z|$ so small that $|\sum_{m+1}^{\infty} \phi(f_n)\, z^n| \le |z^m\, \phi(f_m)|/2$, we obtain

$$|\phi(f(z)| \ge |\phi(f_0) + z^m\, \phi(f_m)| - |z^m \phi(f_m)|/2 = |\phi(f_0)| + |z^m \phi(f_m)|/2,$$

implying $\phi(f_m) = 0$, hence $f_m = 0$. □

Another way of expressing the above result is by saying that for arbitrary $f \in \boldsymbol{H}(G, \mathbb{E})$ with $f'(z) \not\equiv 0$ the function $F(z) = \|f(z)\|$ cannot have a local maximum in G. Using this, we now prove the following important result:

Theorem 70 (Phragmén-Lindelöf Principle) *For $k > 0$, let $S = \{\, z : |z| > \rho_0,\ \alpha < \arg z < \beta\,\}$, with $\rho_0 > 0$, $0 < \beta - \alpha < \pi/k$. For some $f \in \boldsymbol{H}(S, \mathbb{E})$, assume $\|f(z)\| \le c \exp[K|z|^k]$ in S, for sufficiently large $c, K > 0$. Moreover, assume that f is continuous up to the boundary of S and is bounded by a constant C there. Then*

$$\|f(z)\| \le C, \quad z \in S.$$

Proof: Without loss of generality, let the positive real axis bisect the sector S. Then for κ larger than, but sufficiently close to, k and arbitrary $\varepsilon > 0$, the function $\mathrm{e}^{-\varepsilon z^{\kappa}}$ decreases throughout S, and therefore $F(z) = \mathrm{e}^{-\varepsilon z^{\kappa}} f(z)$ tends to zero as $z \to \infty$ in S. For every sufficiently large $\rho > 0$ we conclude that $\|F(z)\|$ is bounded by C, for z on the boundary of the region $S \cap D(0, \rho)$. The Maximum Modulus Principle then implies the same estimate inside the region. Letting $\varepsilon \to 0$ and $\rho \to \infty$, one can complete the proof. □

One should note that we have tacitly allowed that S may be a region on the Riemann surface of the logarithm, i.e., $\beta - \alpha$ may be larger than 2π. The theorem admits various generalizations; for a very useful cohomological version, see *Sibuya* [251].

Exercises:

1. Show that the above theorem does not hold in general if the sector S has opening larger than or equal to π/k.

2. Let f be an $\mathbb{E}$-valued entire function that is bounded in a closed sector $\bar{S} = \{z : \ \alpha \le \arg z \le \beta\}$. What can be said about its order?

3. Compare Exercise 7 on p. 63 to see that entire functions of infinite order exist that are bounded in closed sectors of openings $2\pi - \varepsilon$, with arbitrary small $\varepsilon > 0$.

Appendix C
Functions of a Matrix

There is a beautiful classical theory of what is called a *function of a matrix*, to be distinguished from the term *matrix function.* In this context, one is concerned with interpreting $f(A)$, for a suitable class of $\mathbb{C}$-valued functions f and square matrices A. More generally, one can instead of matrices choose A in a Banach algebra, but we shall not need this here. In the simplest case, f will be a polynomial, say, $f(z) = \sum_0^m f_n z^n$, and then we simply set $f(A) = \sum_0^m f_n A^n$. Similarly, when f is an entire function, one can use the power series expansion of f to define $f(A)$, and we shall do this in the case of $f(z) = \mathrm{e}^z$ in the next section. More interesting are the cases when f is holomorphic in a region $G \neq \mathbb{C}$. It can then be seen that it is crucial to restrict ourselves to matrices with spectrum in G, and then one can define $f(A)$ very elegantly by a generalization of Cauchy's integral formula. A similar representation holds even for multivalued holomorphic functions on a Riemann surface.

For applications in the theory of ODE, it is sufficient to understand, aside from the exponential function, the natural logarithm of a matrix and the closely related matrix powers of a complex variable z. In the following sections we shall present some basic facts concerning these concepts. In particular, we shall show that every invertible matrix has a matrix logarithm which, however, is not uniquely defined. For matrices A that are close to the unit matrix, i.e., $\|I - A\|$ is small, one can define the main branch of $\log A$ by means of the power series expansion of the logarithm, and the same is possible whenever $I - A$ is nilpotent. For general A, however, we have to find a different definition of $\log A$; this will be done in Section C.2, where we shall also show that we can select a unique branch of $\log A$ by requiring its eigenvalues to have imaginary parts in a fixed half-open interval of length 2π. This corresponds to selecting a fixed branch of $\log z$ by restricting the argument of z accordingly.

C.1 Exponential of a Matrix

Given a square matrix A, we define

$$\mathrm{e}^A = \exp[A] = \sum_{n=0}^{\infty} \frac{A^n}{n!},$$

where $A^0 = I$ and $A^n = A \cdot \ldots \cdot A$, n times repeated, for $n \geq 1$. From $\| \sum_m^\mu A^n/(n!) \| \leq \sum_m^\mu \|A\|^n/(n!)$ one can easily conclude convergence of the series, thus *the exponential of a matrix* is always defined. Manipulating with the infinite matrix power series, one can show the following rules:

1. For $A, T \in \mathbb{C}^{\nu \times \nu}$, T invertible, we have
$$\exp[T^{-1} A\, T] = T^{-1} \exp[A]\, T.$$
2. For $\Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_\nu]$ we have
$$\exp[\Lambda] = \operatorname{diag}[\mathrm{e}^{\lambda_1}, \ldots, \mathrm{e}^{\lambda_\nu}].$$
3. For a triangularly blocked matrix A with diagonal blocks A_k, the matrix $\exp[A]$ is likewise blocked and has diagonal blocks $\exp[A_k]$, in the same order.
4. For matrices $A, B \in \mathbb{C}^{\nu \times \nu}$ with $AB = BA$, we have
$$\exp[A + B] = \exp[A]\, \exp[B] \ (\, = \exp[B]\, \exp[A]\ ),$$
but this rule does not hold in general, as is shown in one of the exercises below.
5. For every square matrix A, the inverse of $\exp[A]$ exists and equals $\exp[-A]$.

We conclude from the above set of rules that $\exp[A]$ can theoretically be computed by first transforming A into Jordan form $J = \Lambda + N$ with commuting terms Λ (diagonal) and N (nilpotent), and then computing $\exp[\Lambda]$ and $\exp[N]$; for the second note that the defining series terminates.

Closely related to the exponential of the matrix are powers z^A, $z \in \mathbb{C} \setminus \{0\}$, $A \in \mathbb{C}^{\nu \times \nu}$, since we set

$$z^A = \exp[A \log z].$$

To make this definition unambiguous, we have to select a branch of the multivalued function $\log z$, e.g., by specifying a value for $\arg z$. Whether or not z^A is indeed a multivalued function, and how many branches this function has, depends on the eigenvalues of A as well as its nilpotent part; e.g., if A is diagonalizable and all eigenvalues are integers, then z^A is single-valued, while for A being a nilpotent nonzero matrix the function has infinitely many branches. By termwise differentiation of the defining series one can show that

$$\frac{d}{dz} z^A = A\, z^{A-I} = z^{A-I}\, A,$$

independent of the selected branch of the function.

Exercises: If nothing else is said, let

$$A = \begin{bmatrix} A_{11} & O \\ A_{21} & A_{22} \end{bmatrix}$$

be a constant matrix, lower triangularly blocked as indicated, with square diagonal blocks.

1. Assume that A_{11} and A_{22} have disjoint spectra, i.e., do not have a common eigenvalue. Show existence and uniqueness of a constant matrix T so that

$$z^A = \begin{bmatrix} z^{A_{11}} & O \\ T\, z^{A_{11}} - z^{A_{22}}\, T & z^{A_{22}} \end{bmatrix}.$$

2. For a nilpotent Jordan block N, compute z^N.

3. Show: $\det \exp[A] = \exp[\operatorname{trace} A]$.

4. For arbitrary $\nu \times \nu$ matrices A, B, show that

$$\exp[(A+B)z] = \exp[Az]\, \exp[Bz]$$

holds for every $z \in \mathbb{C}$ if and only if A and B commute.

5. Let $A(z)$ be a square matrix whose entries all are holomorphic in some region $G \subset \mathbb{C}$, so that $A'(z)\, A(z) = A(z)\, A'(z)$ for every $z \in G$. Show

$$\frac{d}{dz} \mathrm{e}^{A(z)} = A'(z)\, \mathrm{e}^{A(z)} = \mathrm{e}^{A(z)}\, A'(z), \quad z \in G.$$

C.2 Logarithms of a Matrix

Every reasonable interpretation for $\log A$ should be a matrix X solving the equation $e^X = A$. This is not a very suitable definition, but it shows that we cannot expect a matrix A to have a logarithm whenever $\det A = 0$. Thus, *in the following we restrict to invertible matrices A*. The most elegant definition of logarithms of matrices is by means of Cauchy's integral formula: Let an invertible matrix A be given, whose eigenvalues λ_k, $1 \le k \le \mu$, necessarily are all nonzero. We define

$$\log A = (2\pi i)^{-1} \int_\gamma \log z \, (zI - A)^{-1} dz, \tag{C.1}$$

where γ is not a single curve, but rather a collection of small circles, positively oriented, one about each eigenvalue λ_k, and none of them encircling the origin. Hence in fact, we do not have one integral but a sum of finitely many, one for each such circle. The radii of these circles should be taken so small that the corresponding closed discs do not intersect, and we may make arbitrary choices for a branch of $\log z$ along each one of the circles. Every matrix $\log A$ computed according to this formula will be called *a branch of* the logarithm of A. Note that we cannot hope for a single-valued function, since the logarithm already is multivalued in the scalar case.

From the above definition one can deduce a list of rules that enable us, in principle, to compute the branches for $\log A$. Note, however, that formulas containing more than one matrix logarithm hold only if, after choosing branches for all but one logarithm, we choose the correct branch for the remaining one.

- For a diagonal invertible matrix $\Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_\nu]$, we have

  $$\log \Lambda = \operatorname{diag}[\log \lambda_1, \ldots, \log \lambda_\nu],$$

  where on the right we choose branches of the logarithms such that $\lambda_j = \lambda_k$ implies $\log \lambda_j = \log \lambda_k$.

- For a nilpotent matrix N, one (the main) branch of the logarithm is

  $$\log(I + N) = \sum_{n \ge 1} \frac{(-1)^{n+1}}{n} N^n.$$

  Observe that the series terminates. Every other branch is obtained from the main one by adding $2k\pi i I$, $k \in \mathbb{Z}$.

- For a Jordan matrix $J = \Lambda + N = \Lambda(I + \Lambda^{-1}N)$, with invertible diagonal Λ and nilpotent N, commuting with Λ, we have

  $$\log J = \log \Lambda + \log(I + \Lambda^{-1}N),$$

provided that the logarithms on the right are computed according to the rules stated above; observe that then the matrices on the right commute!

- For $A = T^{-1} J\, T$, with J in Jordan form, we have

$$\log A = T^{-1} \log(J)\, T.$$

It can be shown that the following identities for logarithms of matrices hold; observe, however, what was said above about formulas containing several logarithms.

1. For every invertible A we have

$$\exp[\log A] = A.$$

   However, for certain A there are X with $\exp[X] = A$ so that no branch of $\log A$ equals X.

2. For $B = T^{-1} A\, T$ we have

$$\log B = T^{-1} (\log A)\, T.$$

3. For A, B commuting and invertible, the rule

$$\log(A\, B) = \log A \; + \log B$$

   holds modulo the following restriction: If branches for any two out of the three logarithms are selected, then the formula is correct provided that we select the branch for the third accordingly.

4. For triangularly blocked invertible matrices A with diagonal blocks A_k, all branches of $\log A$ are likewise blocked and have diagonal blocks $\log A_k$ in the same order.

For our purposes it is important to be able to solve equations of the form $\exp[2\pi i X] = C$, for given invertible C. From what we have said above about logarithms of matrices, one can show that the following is correct:

Theorem 71 *Given any invertible matrix C, there exist matrices M such that $\exp[2\pi i M] = C$; namely, $M = (2\pi i)^{-1} \log C$. If C is triangularly blocked of some type, then so are the matrices M. The eigenvalues of any two such M can only differ by integers, and there is a unique matrix M whose eigenvalues have real parts in the half-open interval $[0, 1)$.*

Exercises: In what follows, consider the matrices

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2\pi i \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

1. Show that $\mathrm{e}^A = I$.
2. Show that the above matrix A is not a branch of $\log I$.
3. For $A = I + N$, N nilpotent, show

$$(zI - A)^{-1} = \sum_{n \geq 0} (z-1)^{-n-1} N^n$$

(observe that the series terminates). Use this to obtain the formula given for the main branch of $\log(I + N)$ from (C.1).

Solutions to the Exercises

Here we give some detailed hints toward the solution of the exercises, in particular to those that are used in later chapters of the book. Exercises left out here require only straightforward computation.

Section 1.1:

1. Since $\boldsymbol{H}(G, \mathbb{C})$ is a vector space over $\mathbb{C}$, it suffices to verify the *subspace* properties.

2. Define $x_0(z) \equiv x_0$, $x_{k+1}(z) = \int_{z_0}^{z} A(u)\, x_k(u)\, du$, $k \geq 0$, and show the following:

   (a) Every $x_k(z)$ is holomorphic in D, $k \geq 0$.

   (b) For every $\tilde{\rho} < \rho$, there exists $K = K(\tilde{\rho})$ such that $\|x_k(z)\| \leq K^k |z - z_0|^k \, \|x_0\| / k!$, $|z - z_0| \leq \tilde{\rho}$, $k \geq 0$.

   (c) The series $x(z) = \sum_0^\infty x_k(z)$ converges *compactly* on D, i.e., uniformly on compact subsets of D, and represents a solution of the initial value problem.

   (d) If $\tilde{x}(z)$ is any other solution of the initial value problem, then for every $\tilde{\rho} < \rho$ there exist $K = K(\tilde{\rho})$ and $M = M(\tilde{\rho})$ such that $\|x(z) - \tilde{x}(z)\| \leq M\, K^k |z - z_0|^k / k!$, $|z - z_0| \leq \tilde{\rho}$, $k \geq 0$; thus for $k \to \infty$ we conclude $x(z) \equiv \tilde{x}(z)$.

3. Proceed as above, estimating on compact sets that are star-shaped with respect to z_0.

4. For $G = \mathbb{C}$, the proof is completed. For every other G, there is a function ϕ, holomorphic in the unit disc D and mapping D bijectively into G. For $\tilde{x}(z) = x(\phi(z))$, $\tilde{A}(z) = \phi'(z)\,A(\phi(z))$, verify that (1.1) is equivalent to $\tilde{x}' = \tilde{A}(z)\,\tilde{x}$, $z \in D$.

6. (a) $a_1(z) = 2z\,(1-z^2)^{-1}$, $a_2(z) = -\mu(1-z^2)^{-1}$.
   (b) $(n+1)(n+2)\,x_{n+2} = [n(n+1)-\mu]\,x_n$, $n \ge 0$. Since all solutions are holomorphic in the unit disc, the power series converges for at least $|z| < 1$. The quotient test, applied to the odd and even part of a solution, shows that the radius of convergence equals 1, except when we have a polynomial solution.
   (c) For m even (odd), let all x_n with odd (even) n be zero. Then for $x_0 = 1$ ($x_1 = 1$), we obtain a unique polynomial of degree m.
   (d) Follows from the recursion formula.

7. (a) Expanding $t_k(z) = \sum_0^\infty t_n^{(k)}(z-z_0)^n$, for $z \in D(z_0,\rho) \subset G$, show by induction with respect to k that $t_n^{(k)}$ depends linearly upon $t_0^{(0)}, \ldots, t_{n+k}^{(0)}$, but is independent of the remaining $t_m^{(0)}$, $m \ge n+k+1$. In particular, verify $t_n^{(k)} = (n+k)!\,t_{n+k}^{(0)}/n! + \tilde{t}$, with $\tilde{t}$ depending only upon $t_0^{(0)}, \ldots, t_{n+k-1}^{(0)}$. Next, show by induction that one can choose the vectors $t_0^{(0)}, \ldots, t_k^{(0)}$ so that the matrix T_k with rows $t_0^{(0)}, \ldots, t_0^{(k)}$ has maximal rank.
   (b) Show $T'(z) + T(z)\,A(z) = B(z)\,T(z)$ on D. Also, use Theorem 1 (p. 4) to conclude that $x(z) = T^{-1}(z)\,\tilde{x}(z)$, which is defined and holomorphic on D, can be holomorphically continued into all of the region G.

Section 1.2:

2. Compare Exercise 5 on p. 239 for how to differentiate the exponential of a matrix.

3. As above.

5. Set $A(z) = X'(z)\,X^{-1}(z)$.

6. Differentiate the identity $X(z)\,X^{-1}(z) \equiv I$ to obtain $[X^{-1}(z)]' = -X^{-1}(z)\,X'(z)\,X^{-1}(z)$, $z \in G$.

7. For a fundamental solution $\tilde{X}(z)$ of (1.1), conclude $X(z) = \tilde{X}(z)\,C$, with a constant matrix of suitable size. Use this and elementary results on the rank to see that $X(z)$ has the same rank as C.

8. Take $x_{\mu+1}(z), \ldots, x_\nu(z)$ constant so that the said determinant is nonzero at some selected point $z_0 \in G$, and then conclude by holomorphy that it is nonzero on some disc around z_0. The rest should be straightforward.

Section 1.3:

1. Compare Section C.1 for how to differentiate the matrix z^M.

2. Take $M = \lambda I$, hence $z^M = z^\lambda I$, and distinguish the cases $\lambda \in \mathbb{Z}$, $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

4. Consider $X(z) = (z-z_1)^{M_1} \cdot \ldots \cdot (z-z_\mu)^{M_\mu}$ and use the commutativity of the M_k.

5. Compare Exercise 5 on p. 7.

6. The proof of the statement concerning monodromy factors is straightforward. Concerning the monodromy matrices, conclude that for some constant invertible matrix C we must have that $z^{M_1} C\, z^{-M_2}$ is single-valued. Then, show the same for each M_k replaced by its Jordan canonical form J_k, and conclude that this implies $\mathrm{e}^{2\pi i J_1} C = C \mathrm{e}^{2\pi i J_2}$. Finally, relate the eigenvalues of $\mathrm{e}^{2\pi i J_k}$ and J_k.

7. Use properties of logarithms of matrices, as stated in Section C.2.

8. Show existence of a fundamental solution $X(z)$ with a monodromy matrix in (lower triangular) Jordan form, and consider the last column of $X(z)$.

9. Use the previous exercise, together with Exercise 5 on p. 4.

Section 1.4:

1. $(n+1)x_{n+1} = b_n + \sum_{m=0}^{n} A_{n-m}\, x_m$, $n \geq 0$; thus, x_0 can be chosen arbitrarily, while the other coefficients are determined.

2. For $b(z) = \sum_{n=0}^{d} b_n\, z^n$, $x(z) = \sum_{n=0}^{d} b_n\, z^n$, we have (1.10) if and only if $(n+1)x_{n+1} = b_n + A\, x_n$, $0 \leq n \leq d-1$, and $0 = b_d + A\, x_d$. In order that the last equation can be satisfied for arbitrary b_d, we have to assume that A is invertible. If this is so, then the previous equations can all be solved for x_n. Hence, the polynomial solution even is unique!

4. For $x(z)$ as in (1.11), let $x(z \exp[2\pi i])$ denote its holomorphic continuation about the origin in the positive sense. Show

$$\begin{aligned} x(z\mathrm{e}^{2\pi i}) &= X(z\mathrm{e}^{2\pi i}) \left[c + \tilde{c} + \int_{z_0}^{z} X^{-1}(u\mathrm{e}^{2\pi i})\, b(u)\, du\right] \\ &= X(z) \left[\mathrm{e}^{2\pi i M}(c + \tilde{c}) + \int_{z_0}^{z} X^{-1}(u)\, b(u)\, du\right], \\ \tilde{c} &= \int_{z_0}^{z_0 \mathrm{e}^{2\pi i}} X^{-1}(u)\, b(u)\, du, \end{aligned}$$

and compare this to (1.11). A sufficient condition is that the left-hand side of (1.12) vanishes only for $c = 0$.

Section 1.5:

3. Verify that it suffices to treat the case $\mu = 2$. Then, verify that the diagonal blocks of the monodromy factor for $X(z)$ are equal to C_1, C_2, and let the unknown block below the diagonal be denoted by B. Next, show $Y_{21}(u \exp[2\pi i]) = Y_{21}(u)\, C_1$ and use this to conclude

$$X_{21}(z\mathrm{e}^{2\pi i}) = X_{22}(z) \left[C_2(C_{21} + \tilde{C}_{21}) + \int_{z_0}^{z} X_{22}^{-1}(u)\, Y_{21}(u)\, du\, C_1\right],$$

with $\tilde{C}_{21}$ given by a definite integral. From this identity, conclude $B = C_2(C_{21} + \tilde{C}_{21}) - C_{21}\, C_1$.

4. Use Exercise 8 on p. 7 and Theorem 4.

Section 1.6:

4. For $z_0 = (\rho + \varepsilon) \exp[i\tau]$, $\alpha \le \tau \le \beta$, observe $X(z) = X(z_0) + \int_{z_0}^{z} A(w)\, X(w)\, dw$. Setting $n(x) = \|X(x^{1/r}\mathrm{e}^{i\tau})\|$, conclude $n(x) \le c + a \int_{x_0}^{x} n(t)\, dt$, for a sufficiently large $c > 0$. Then, show by induction

$$n(x) \le c \sum_{k=0}^{m} \frac{[a(x - x_0)]^k}{k!} + a^{m+1} \int_{x_0}^{x} \frac{(x-t)^m}{m!} n(t)\, dt$$

and let $m \to \infty$. Finally, discuss the dependence of c upon z_0.

5. Consider $X(z) = \exp[A\, z^r/r]$, with A not nilpotent.

6. Consider $X(z) = \exp[N\, z^r/r]$, with N nilpotent, but not the zero matrix. For $\nu = 1$, observe that every solution $x(z)$ can be represented in the form $x(z) = c \exp[b(z)]$, with $b'(z) = A(z)$.

8. Consider $G = \mathbb{C}$ and $X(z) = f(z)\, I$, with an entire function having infinitely many zeros, e.g., $f(z) = \sin z$. Try to generalize this to arbitrary regions G.

Section 2.1:

1. (a) Conclude that $B(z) = A(z)\, P_N(z) - z P_N'(z) - P_N(z)\, A_0$ is holomorphic in $D(0,\rho)$ and vanishes at the origin of order $N+1$.
   (b) For N sufficiently large, let $\tilde S_0(z) = \int_0^z B(u)\, u^{A_0 - I} du\, z^{-A_0}$, and $\tilde S_k(z) = \int_0^z A(u)\, \tilde S_{k-1}(u)\, u^{A_0-I} du\, z^{-A_0}$, $k \ge 1$. Show that each $\tilde S_k(z)$ is holomorphic in $D(0,\rho)$ and vanishes at the origin of order $N+1$.
   (c) For N as above and $\tilde\rho < \rho$, show the existence of $a \in \mathbb{R}$, $c, K > 0$, with K and a independent of N, such that (with $\arg z \in [0, 2\pi]$) $\|\tilde S_k(z)\, z^{A_0}\| \le c\,(K/(N-a))^k |z|^{N-a}$, $k \ge 0$, $|z| \le \tilde\rho$. Use this and the Maximum Principle to show, for sufficiently large N, uniform convergence of $\tilde S(z) = \sum_0^\infty \tilde S_k(z)$, $|z| \le \tilde\rho$, and conclude that $\tilde S(z)$ is holomorphic in $D(0,\rho)$ and vanishes at the origin of $N+1$st order.
   (d) Show that $\tilde X(z) = \tilde S(z)\, z^{A_0}$ is a solution of (2.4).

3. For existence, use the previous exercise and the form of z^J. The recursion formula is $[(n+\mu)I - A_0]\, s_n = \sum_{m=0}^{n-1} A_{n-m}\, s_m, \quad n \ge 1$.

4. Follows from the previous exercise.

5. Compute z^{A_0}.

Section 2.2:

1. Let λ_k denote the two distinct eigenvalues of A, and define $\beta = \mu_1 - \mu_2$, $\lambda = \lambda_2 - \lambda_1$, $\alpha = (a - \lambda_1)\,\beta/\lambda$. Then one Floquet solution is given by $x_1(z) = \mathrm{e}^{\lambda_1 z}\, z^{\mu_1}\, F(\alpha; \beta; \lambda z)$, $x_2(z) = c\,\mathrm{e}^{\lambda_1 z}\, z^{\mu_1 + 1}\, (1+\beta)^{-1}\, F(1+\alpha; 2+\beta; \lambda z)$. The second one is given similarly.

2. For λ_1, β as above and $\omega^2 = 4(\lambda_1 - a)\beta$, resp. $\mu = (\mu_1 + \mu_2 + 1)/2$, one solution is given by $x_1(z) = \mathrm{e}^{\lambda_1 z}\, z^{\mu}\, \Gamma(\beta)\, (\omega/2)^{1-\beta}\, J_{\beta-1}(\omega z^{1/2})$, $x_2(z) = \mathrm{e}^{\lambda_1 z}\, z^{\mu} c\, \Gamma(1+\beta)\, (\omega/2)^{-\beta-1}\, J_{\beta+1}(\omega z^{1/2})$.

4. Show that the transformation $y(z) = \mathrm{e}^z \tilde y(-z)$ carries the confluent hypergeometric equation into one of the same kind, but with α replaced by $\beta - \alpha$, and use that the equation has exactly one solution in form of a power series with constant term equal to 1.

5. Use the *Beta Integral* (B.11).

7. For $z \in \mathbb{C}$, show

   $\frac{d}{dz} F(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_m; z) =$

$$\left[\prod_{j=1}^m \alpha_j/\beta_j\right] F(\alpha_1+1,\ldots,\alpha_m+1;\beta_1+1,\ldots,\beta_m+1;z).$$

$$(\delta+\alpha_1)\,F(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_m;z) = \alpha_1\,F(\alpha_1+1,\alpha_2\ldots,\alpha_m;\beta_1,\ldots,\beta_m;z).$$

$$(\delta+\beta_1)\,F(\alpha_1,\ldots,\alpha_m;\beta_1+1,\beta_2,\ldots,\beta_m;z) = \beta_1\,F(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_m;z).$$

From these identities, the differential equation follows easily.

Section 2.3:

2. Use analogous arguments as in Exercise 7 on p. 25.

4. Expand $(1-zt)^{-\alpha}$, for $|z|<1$, into its power series, justify termwise integration, and use the *Beta Integral* (B.11).

5. Use Stirling's formula (B.12) to show convergence of the series; then, restrict to $\mathrm{Re}\,\gamma > \mathrm{Re}\,\beta > 0 \geq \mathrm{Re}\,\alpha$ and use (2.10), and finally, use holomorphy of both sides of (2.11) in all variables α, β, γ to remove this restriction.

6. Show that the right-hand side solves the hypergeometic equation. Then show that the equation has only one power series solution with constant term equal to 1.

Section 2.4:

1. Observe that a fundamental solution is $X(z)=z^K\,z^M$.

2. Note that in (2.18) we have to set $A_0=B_0=B$, $A_1=A$, and the remaining A_n equal to 0. For $n=1$, use $T_0=I$ to see that the equation becomes solvable if we take
$$B_1 = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix},$$
and then
$$T_1 = \begin{bmatrix} a & b/2 \\ 0 & d \end{bmatrix}$$
follows, if we select the undetermined entry in the $(2,1)$ position equal to zero. For $n\geq 2$, the recursion equations following from (2.18) are
$$\begin{aligned} n\,t_{11}^{(n)} &= a\,t_{11}^{(n-1)} + b\,t_{21}^{(n-1)} - c\,t_{12}^{(n-1)}, \\ (n-1)\,t_{21}^{(n)} &= c\,t_{11}^{(n-1)} + d\,t_{21}^{(n-1)} - c\,t_{22}^{(n-1)}, \\ (n+1)\,t_{12}^{(n)} &= a\,t_{12}^{(n-1)} + b\,t_{22}^{(n-1)}, \\ n\,t_{22}^{(n)} &= c\,t_{12}^{(n-1)} + d\,t_{22}^{(n-1)}. \end{aligned}$$

Solving the system $z\tilde{x}' = B(z)\,\tilde{x}$, one finds as fundamental solution

$$\tilde{X}(z) = \begin{bmatrix} 1 & 0 \\ zc\log z & z \end{bmatrix} = z^K\, z^M,$$

with

$$K = \operatorname{diag}[0,1], \qquad M = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}.$$

Section 2.5:

3. Set $p_0(x) = [x]_\nu - \sum_{k=1}^{\nu} b_0^{(k)}\,[x]_{\nu-k}$, $p_n(x) = \sum_{k=1}^{\nu} b_n^{(k)}\,[x]_{\nu-k}$, $n \ge 1$. Then
$$p_0(\mu+n)\,y_n = \sum_{m=0}^{n-1} p_{n-m}(\mu+m)\,y_m, \quad n \ge 1.$$

4. For part (c), differentiate the inhomogeneous ODE of (a) with respect to w.

Section 3.1:

1. Estimate t^y by one.

2. Use the previous exercise.

3. Find the recursion relation for the coefficients of the inverse, and estimate similarly to the proof of the Splitting Lemma in Section 3.2.

4. Use Proposition 5 (p. 40), and split $\hat{T}_1(z) = P(z^{-1})(I + O(z^{-\mu}))$, with a matrix polynomial P and sufficiently large $\mu \in \mathbb{N}$.

6. Use Stirling's formula (p. 229).

Section 3.2:

1. $T_n^{(12)} = b\,(1+a)_{n-1}$, $n \ge 1$. Hence, the rate of growth roughly is as $n!$, except when the sequence terminates.

4. Use Theorem 8 (p. 44) and the previous exercise.

Section 3.3:

1. Let $c_1, \ldots, c_\nu$ denote the elements in the first row of C, and denote the characteristic polynomial by $p(\lambda, c_1, \ldots, c_\nu)$. Expansion with respect to the last column shows $p(\lambda, c_1, \ldots, c_\nu) = (-1)^{\nu+1} c_\nu - \lambda p(\lambda, c_1, \ldots, c_{\nu-1})$, from which follows

$$p(\lambda, c_1, \ldots, c_\nu) = (-1)^\nu [\lambda^\nu - \sum_{j=1}^{\nu} c_j \lambda^{\nu-j}].$$

2. The only case for $A_0 \neq 0$ but nilpotent is

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

   If the $(1,2)$-element of A_1 is nonzero, use $T(z) = \operatorname{diag}[1, z^{1/2}]$; otherwise use $T(z) = \operatorname{diag}[1, z]$.

3. For any $T(z)$ as in (3.12), make suitable assumptions on the coefficients $A_1, A_2, \ldots$ so that this shearing transformation produces a system whose leading term has several eigenvalues.

4. From the results of this section, show existence of a q-meromorphic transformation for which the transformed system, after the change of variable $z = w^q$, has Poincaré rank strictly smaller than qr, and then use Exercise 4 on p. 15.

Section 3.4:

1. Set $B(z) = z^r \sum_{n=0}^{r} A_n z^{-n}$. *Assuming* that $\hat{T}(z)$ commutes with $B(z)$, the problem reduces to solving $z\, T'(z) = \hat{C}(z)\, \hat{T}(z)$, $\hat{C}(z) = \hat{A}(z) - B(z) = \sum_{n=1}^{\infty} C_n z^{-n}$. This leads to a simple recursion equation for T_n.

2. The pole order is r.

Section 3.5:

1. Use the exponential shift $x = \exp[q(z)]$, $z\, q'(z) = z^r \sum_{n=0}^{r-1} A_n z^{-n}$.

2. In case (a), we have $q = 1$. *Theoretically*, an HLFFS is obtained in two steps: First we apply the Splitting Lemma, decoupling the system into two one-dimensional systems. Then we apply Theorem 11 (p. 52) to both of these systems. From Exercise 1 on p. 55 we conclude that we are able to take $N = 1$ and $M = r - 1 = 0$. This shows that there is an HLFFS for which the transformed system has a coefficient matrix

of the form $B(z) = z A_0 + \mathrm{diag}\,[a, d]$, and then $\hat{F}(z) = I + \sum_1 F_n\, z^{-n}$ can be computed directly as follows: Denoting the entries in the first column of F_n by f_n and g_n, they satisfy the recursions

$$n\, f_n = -b\, g_n, \;\; (\lambda_1 - \lambda_2)\, g_{n+1} = (n + d - a)\, g_n + c\, f_n, \quad n \ge 1,$$

with $g_1 = c/(\lambda_1 - \lambda_2)$. Setting $\alpha + \beta = d - a$, $\alpha\, \beta = -b\, c$, we obtain

$$\left.\begin{array}{rcl} f_n & = & \frac{(\alpha)_n\, (\beta)_n}{(\lambda_1 - \lambda_2)^n\, n!} \\ g_n & = & c\, \frac{(1+\alpha)_{n-1}\, (1+\beta)_{n-1}}{(\lambda_1 - \lambda_2)^n\, (n-1)!} \end{array}\right\}, \quad n \ge 1.$$

Similarly, one can obtain the other two entries of F_n.

In case (b), use a scalar exponential shift to eliminate λ, then use a constant transformation T that commutes with A_0 and produces a new matrix A_1 for which the (new) element c vanishes. Then, use the transformation

$$x = \mathrm{diag}\,[1, z^1/2] \begin{bmatrix} b^{1/2} & -b^{1/2} \\ 1 & 1 \end{bmatrix} \tilde{x}$$

and the change of variable $z = w^2$ to obtain a system of the form treated in case (a).

Section 4.1:

1. Note that by definition of sectorial regions it suffices to prove the statement for sectors.

2. Use the previous exercise to see that it suffices to deal with $k = 1$ and $\tau = 0$, in which case the inequalities describe a circle with center on the positive real axis, touching the origin, resp. the right half-plane in case of $c = 0$.

3. Let $I = (a, b)$; then the union has opening $\alpha + b - a$ and bisecting direction $(b + a)/2$.

Section 4.2:

1. Use $[f(z) - f(0)]\, z^{-1} - f'(0) = z^{-1} \int_0^z [f'(u) - f'(0)]\, du$.

2. Conclude that differentiability at the origin is equivalent to $f(z) = f(0) + z\, f'(0) + z\, h(z)$, with $h(z)$ continuous at the origin and $h(0) = 0$. Then, use Cauchy's integral formula for $f'(z)$.

3. Use the fact that, according to the previous exercises, differentiability at the origin is equivalent to continuity of the derivative at the origin.

4. Justify termwise integration of the power series expansion of e^{zt}.

Section 4.3:

6. Proceed as in Exercise 2 on p. 41 to show that $\mathbb{E}\,[[z]]_s$ is closed with respect to multiplication. Moreover, use (B.13) to derive that $\mathbb{E}\,[[z]]_s$ is closed with respect to derivation.

7. Compare Exercise 3 on p. 41.

8. Use (B.13) (p. 231).

Section 4.4:

1. Observe $1 - \mathrm{e}^{-z} = \int_0^z \mathrm{e}^{-w}\,dw$ and estimate the integral.

5. Use the previous two exercises.

6. Use Proposition 8 (p. 66).

Section 4.5:

2. Use Exercise 1.

3. Use (B.13) (p. 231).

5. Observe $z^m\,f(z) - \sum_{n=0}^{N-1} f_{n-m}\,z^n = z^m\,(f(z) - \sum_{n=-m}^{N-m-1} f_n\,z^n)$ and use (B.13) (p. 231).

6. To show that every element of $\boldsymbol{A}_{s,m}(G,\mathbb{C})$ is invertible, observe that the number of zeros of $f(z)$ in an arbitrary closed subsector of G is finite.

7. Take any $\hat{f} \notin \mathbb{E}\,[[z]]_s$ $\forall s \geq 0$, and observe *Ritt's theorem.*

Section 4.6:

2. For (a), use the same arguments as in the proof of Proposition 7 (p. 65) to show $\|g_n\| \leq c_n$ for every $n \geq 0$. For (b), use the integral representation for the Gamma function to show

$$z^N r_f(z,N) = z^{-k} \int_0^{\infty(d)} u^N r_g(u,N)\,\mathrm{e}^{-(u/z)^k}\,du^k$$

and estimate as usual.

4. For $x \geq 0$, find the maximum of $x^n\,\mathrm{e}^{-x}$ and use Stirling's formula.

6. Abbreviate $s = 1/k$. By assumption there exist $c, K > 0$ so that $\|f(z)\| \le c\,(|z|K)^n\,\Gamma(1+sn)$ for every $n \ge 0$ and $z \in \bar{S}$. According to Stirling's formula the same holds, for different constants c, K, with $\Gamma(1+sn)$ replaced by n^{sn}. For small $|z|$, the term $(|z|K)^n\,n^{sn}$ first decreases, then increases with respect to n and is approximately minimal for n with $|z|K\,\mathrm{e}^s n^s \le 1 < |z|K\,\mathrm{e}^s(n+1)^s$. For this n we have $(|z|K)^n\,n^{sn} \le \mathrm{e}^{-sn} \le \tilde{c}\mathrm{e}^{-\tilde{K}|z|^{-k}}$, with sufficiently large constants $\tilde{c}, \tilde{K}$, independent of z.

7. Use the previous exercise.

Section 4.7:

2. Let $\tilde{G} = \{z \in G : \; z\mathrm{e}^{2\pi i} \in G\}$. Conclude that $f(z) - f(z\mathrm{e}^{2\pi i}) \cong_s \hat{0}$ in $\tilde{G}$ and then use *Watson's Lemma* and Proposition 7 (p. 65).

Section 5.1:

1. Make a change of variable $(u/z)^k = x$ in (5.1). The formula holds whenever the integral exists, which is certainly the case for $\lambda = 0$.

2. Make a change of variable $(u/z)^k = x$ in (5.1) and estimate f.

3. Substitute $u^k = a^k + t^k$, conclude that $\tilde{f}(t) = f((a^k + t^k)^{1/k})$ is of exponential growth at most k on the line $\arg t = \tau$, and use the previous exercise.

Section 5.2:

1. Use (B.10) (p. 228).

2. Estimate the integral for $g = \mathcal{B}_k f$ in terms of the length of the path of integration times maximum of the integrand, occurring along the circular part of $\gamma(\tau)$, to obtain $\|g(u)\| \le c\exp[K\rho^{\kappa_1} + (|u|/\rho)^k]$, with ρ being the radius of the circular part. Then, observe that ρ can be selected, depending upon $|u|$, so that the right-hand side is minimal.

3. Make a change of variable $z^{-k} = w$, both in (5.4) and (5.3).

Section 5.3:

2. For real $t > 0$, define

$$h^{\pm}(z) = \int_0^{\infty(\pm\tau)} f(u)\,\mathrm{e}^{-uz}\,\frac{du}{u-t},$$

with sufficiently small $\tau > 0$. Use Theorem 22 (p. 79) to show that $h^{\pm}(z)$ is holomorphic in a region $G^{\pm}$ containing the straight line $z(y) = x_0 + iy$, with sufficiently large x_0 (fixed) and $-\infty < y < \infty$, and $h^{+}(z(y))$ (resp. $h^{-}(z(y))$ tends to zero as $y \to -\infty$ (resp. $y \to +\infty$). For $z(y)$ as above, check that $h^{\pm}(z(y))$ can be represented by an integral as above, but integration along the positive real axis, with a small detour about t, say, along a half-circle, in the upper, resp. lower, half-plane. Using Cauchy's integral formula, show

$$h^{-}(z(y)) = h^{+}(z(y)) + 2\pi i f(t)\, \mathrm{e}^{-tz(y)}.$$

From this "connection relation," conclude

$$\lim_{y\to\pm\infty} h^{\pm}(z(y))\mathrm{e}^{tz(y)} = \mp 2\pi i\, f(t).$$

Then, observe that the left-hand side of (5.5) can be seen to equal $(2\pi i)^{-1}\,[h^{+}(z(-y)) - h^{+}(z(y))]$. From these identities, conclude (5.5).

Section 5.4:

1. Observe $\sum_{n=1}^{N-1}(w^s/z)^n = (w^{sN}\, z^{1-N} - w^s)\,(w^s - z)^{-1}$, and use (B.10) (p. 228) and (B.14) (p. 232).

2. Observe (B.10) (p. 228). Can you show that the asymptotic expansion is of Gevrey order s by estimating the integral representation for $E_s(z;N)$?

Section 5.5:

1. Compute the moment function corresponding to $e(z;\alpha)$ and define $E(z;\alpha)$ analogous to (5.12). Then, derive the integral representation

$$E(z;\alpha) = \frac{1}{2\pi i}\int_\gamma w^{-\alpha} E(w)\frac{dw}{w - z},$$

for $|z| < 1$, and discuss the behavior for $z \to \infty$.

3. Analyze the behavior of $E(z;\alpha)$ as $z \to \infty$, following the same arguments used in the proof of Lemma 6 (p. 84), and then proceed as in the proof of Theorem 26 (p. 84).

Section 5.6:

2. Show the integral representation

$$k(z) = \int_0^{\infty(\tau)} e_1(u/z)\, E_2(u)\, \frac{du}{u},$$

with $2k|\tau| < \pi$, and discuss convergence of the integral, using the behavior of $E_2(u)$ as u tends to infinity.

3. Use the fact that the moment functions $m_j(u/p)$ correspond to the kernels $p\,e_j(z^p)$, which are of order kp. Moreover, show

$$p\,k(z) = \sum_{j=0}^{p-1} k_p([z\,\mathrm{e}^{2j\pi i}]^{1/p}), \quad 0 < \arg z < 2\pi.$$

Section 5.8:

3. Use Lemma 7 (p. 94).

4. Stirling's formula (B.12) (p. 229) implies that the radius of convergence equals 1. Holomorphic continuation follows from Exercise 3 on p. 90.

Section 6.1:

1. Show the functional equations $g(z) = \sum_{\nu=0}^{m-1} z^{2^\nu} + g(z^{2^m})$, $m \geq 1$. Use this to show that $g(r)$ cannot have a limit as $r \to 1-$. Then, conclude that $g(z)$ cannot have a radial limit at any one of the points $z_{jm} = \exp[2j\pi i 2^{-m}]$, for arbitrary $j \in \mathbb{N}$.

2. If $\mathbb{E}$ contains at least one $a \neq 0$, multiply all the terms of the power series $\hat{f}(z)$ by a.

Section 6.2:

1. Observe $(\mathcal{S}\,(\hat{\mathcal{B}}_k \hat{f}))(u) = (1-u)^{-1}$.

5. Use Exercises 4–7, p. 63.

6. For part (a), show $(1-4z)^{-1/2} = \sum_{n=0}^{\infty} \Gamma(1+2n)\,[\Gamma(1+n)]^{-2}\,z^n$, in the disc $|z| < 1/4$.

Section 6.3:

1. Show that $\sum (e-a)^n$ is a Cauchy sequence, and hence defines some $b \in \mathbb{E}$. Then, verify $a\,b = b\,a = e$.

2. Show first that it suffices to consider the case of $f_0 = e$. Then, conclude for every closed subsector $\bar{S}$ of G with sufficiently small radius that $\|e - f(z)\| < 1$ for every $z \in \bar{S}$, and apply the previous exercise.

3. Compare Exercise 3 on p. 72.

5. Apply Theorem 35, with $\mathbb{F} = \mathbb{C}$, and $\hat{T} = \sum T_n\,z^n$ with $T_n = 0$ for $n \geq 1$, and $T_0\,f = f(\omega)$ for every $f \in \mathbb{E}$.

Section 6.4:

1. The series in Nos. 1, 2, and 5 are k-summable. The one in No. 3 converges for $k \geq 1$, and hence is k-summable in this case, while for $k < 1$, there are infinitely many singular directions. The one in No. 6 is 1-summable.

2. Proceed as in the proof of Lemma 10 part (b).

3. Verify directly that $f(z;\lambda) \cong_1 \hat{f}(z;\lambda)$ in $|d - \arg z| < \pi$; to do so, proceed as in the proof of Theorem 22 (p. 79).

4. Observe $\hat{f}(z;\lambda) = \sum_0^{m-1} \Gamma(\lambda+n)\, z^n + z^m \sum_0^\infty \Gamma(\lambda+m+n)\, z^n$, for sufficiently large integer m, and use the previous exercise.

5. Let $g = \mathcal{S}\,(\hat{\mathcal{B}}_1 \hat{f})$, $g(\cdot;\alpha,\beta) = \mathcal{S}\,(\hat{\mathcal{B}}_1 \hat{f}(\cdot;\alpha,\beta))$. For Re $\beta >$ Re $\alpha > 0$, use the Beta Integral (p. 229) to show

$$g(u;\alpha,\beta) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)} \int_0^1 (1-x)^{\beta-\alpha-1}\, x^{\alpha-1}\, g(xu)\, dx.$$

Use this to conclude $\hat{f}(z;\alpha,\beta) \in \mathbb{E}\,\{z\}_1$. To remove the restrictions on α and β, use termwise integration resp. differentiation of $\hat{f}(z;\alpha,\beta)$.

Section 6.5:

1. Observe that the geometric series is contained in $\mathbb{C}\,\{z\}$, for every $k > 0$.

2. Assume $f(z) \cong_{1/k} \hat{f}(z)$ in a sectorial region of opening larger than π/k, and let $g = \mathcal{L}_k(T^- f)$, $\hat{g} = \hat{\mathcal{L}}_k(\hat{T}^- \hat{f})$. Conclude that $g(z) \cong_{1/k} \hat{g}(z)$ in another sectorial region of opening larger than π/k. This shows that $(\Gamma(1+n/k)/m(n))$ is a summability factor; the other case follows similarly.

3. Follows from the previous exercise.

4. Use the previous exercise and Theorem 31 (p. 94).

5. Verify that $f(z) \cong_{1/k} \hat{f}(z)$ in G implies $f(\lambda z) \cong_{1/k} \hat{f}(\lambda z)$ in a corresponding sectorial region $\tilde{G}$ of the same opening.

6. For $\varepsilon = \exp[2\pi i/p]$ and $\hat{f}(z) = \sum f_n\, z^n$, observe $\sum_{j=0}^{p-1} \hat{f}(z\,\varepsilon^j) = p \sum f_{pn}\, z^{pn}$, and use the previous exercise.

7. Use termwise integration resp. differentiation!

8. Use Exercise 1, and compare Exercise 1 on p. 99.

9. Use Theorem 37 (p. 106).

Section 6.6:

4. Letting $r_n(z)$ denote the said Laplace transform, show first

$$r_{n+1}(z) = r_n(z) - \frac{\omega}{z+\omega}\, r_n(z\omega/[z+\omega]), \quad n \geq 1.$$

6. Show first that a power series in powers of z^p belongs to $\mathbb{E}\{z\}_{k,d}$ if and only if it is in $\mathbb{E}\{z\}_{k,d_j}$, for $d_j = d + 2j\pi/p$.

Section 7.1:

2. Show that it suffices to restrict to $|a| < \varepsilon$.

Section 7.3:

1. Use Theorem 34 (p. 103), recalling that the notion of singular rays implies that $d_0 + 2k\pi$ are also singular, for every $k \in \mathbb{Z}$.

2. $2\,\hat{f}(z) = \sum \Gamma(1+n/(2k))\, z^n + \sum \Gamma(1+n/(2k))\,(-1)^n\, z^n$.

Section 7.4:

2. For large z and suitable path of integration, write the integral in the form

$$f_j(1/z) = z^\lambda e^{p(z)} \int_{\infty(2j\pi/r)}^{z} e^{-p(u)-\lambda \log u}\,(p'(u) + \lambda/u)\,\tilde{g}(1/u)\,du,$$

with a function $\tilde{g}$ analytic near the origin. Then, verify that one can always choose a path of integration along which the equation $t = p(u) + \lambda \log u$ has a unique solution $u = \psi(t)$. Substitute the integral accordingly and estimate.

4. Use Proposition 18.

Section 8.1:

1. Recall that $t_n^{(m)}, b_n^{(m)} \leq K^n$ for every $n, m \geq 1$ (with suitably large $K > 0$).

2. $(1-bx)\,d_{m+1}(x) = bx\,(d_m(x)+\tilde{d}_m(x))$, $(1-ax)\,\tilde{d}_{m+1}(x) = ax\,d_m(x)$, for every $m \geq 0$.

Section 8.2:

1. Use Proposition 26 (p. 217).

2. Use the previous exercise.

3. Compare Proposition 13 (p. 105).

Section 8.3:

1. Set $n = n_0$ and use Lemma 24 (p. 212) to conclude $T_{n_0} = 0$; then increase n_0. For $r = 0$, $A_0 + n\,I$ and B_0 have to have disjoint spectrum, for every $n \geq n_0$.

2. Block rows, resp. columns, of $\hat{T}(z)$ according to the block structure of $\hat{A}(z)$, resp. $\hat{B}(z)$. Then, use the previous exercise and invertibility of $\hat{T}(z)$ to show that to every j there corresponds a unique k so that $\lambda_j = \tilde{\lambda}_k$. Permuting columns of $\hat{T}(z)$, conclude that one may assume this to happen for $k = j$, so $\hat{T}(z)$ diagonally blocked. Use invertibility once more to conclude that then $\tilde{s}_k = s_k$. For the convergence proof, use Exercise 3 on p. 128.

3. Permute rows and columns of $\hat{B}(z)$, or of $\hat{A}(z)$, so that the leading terms agree, which can be done according to the previous exercise. Then in particular both matrices have the same block structure.

Section 8.4:

1. Observe that, owing to commutativity, (3.5) coincides with the recursion formula for T_n.

2. Show that it suffices to consider $\Lambda = \operatorname{diag}[\lambda_1 I_{s_1}, \ldots, \lambda_\mu I_{s_\mu}]$ with distinct λ_m, and show that then N, hence: A, is diagonally blocked in the block structure of Λ. Next, use Lemma 24 (p. 212) to show that B commutes with A if and only if it is likewise diagonally blocked, so B certainly commutes with Λ.

3. Use the series representation. In particular, conclude that $T(z)$ commutes with $\hat{A}(z)$.

4. Use the first exercise to reduce the problem to one where $A_n = 0$ for $n \geq r+1$. Then, use a constant transformation to put A_0 into Jordan canonical form. Next, use $T(z)$ as in the previous exercise to remove its nilpotent part, and conclude from above that then all other coefficients are diagonally blocked in the block structure of A_0. Thus, it suffices to continue with one such block. Repeating these arguments, conclude that every elementary system (3.3) can be transformed into

one where the coefficient matrix has the desired form, except for the conditions upon Re $b_m(0)$. This can then be arranged using shearing transformations.

6. First, show that for a *scalar* formal meromorphic series $\hat{t}(z)$ the equation $z\,\hat{t}'(z) =$ constant implies $\hat{t}(z) =$ constant. Then compute $N_1\,\hat{T}(z) - \hat{T}(z)\,N_2$.

Section 8.5:

1. Build a vector x by rearranging the elements of X.

2. Holomorphicity follows from the results in Chapter 1. For the determinant, show that for an arbitrary fundamental solution $X_2(z)$ of $z\,x' = A_2(z)\,x$, the matrix $X(z)\,X_2(z)$ is a solution of $z\,x' = A_1(z)\,x$, and use Proposition 1 (p. 6).

3. Use the previous exercise.

Section 9.1:

2. For a matrix $C = E_{jm}$ with a 1 in position (j, m) and 0's elsewhere, use the previous exercise to see that $X_1(z)\,E_{jm}\,X_2^{-1}(z)$ is a solution. Check that all these solutions are linearly independent, and use Theorem 2 (p. 6).

4. For (d), show $(n, m) \in \mathrm{Supp}_{j,n_1}$ if and only if (9.2) holds in $S'_j = S_j \cap \ldots \cap S_{j+n_1}$. Use this to conclude that C has the required form if and only if $Y(z)\,(I + C)\,Y^{-1}(z) \cong_{1/k} I$ in S'_j. Then, use results from Section 4.5. For (g), enumerate the data pairs in such a way that (9.2) holds for $d_{j-1} + \pi/(2k) < \arg z < d_j + \pi/(2k)$ if and only if $n < m$, so that $\mathbb{G}_{j,n_1}$ consists of upper triangularly blocked matrices. Then, consider blocks directly above the diagonal first, next, treat those in the next superdiagonal, etc.

Section 9.2:

1. Consider the identities obtained for the blocks of D, C_+, and C_- in a suitable order.

2. For the product on the right, first consider the blocks directly below the diagonal, then in the next subdiagonal, and so forth. Also compare this to Exercise 4 on p. 142.

3. Observe that for integer values of k the number j_0 always is even. Then show that, after a suitable renumeration of the data pairs, i.e., a permutation of the columns of the HLFFS, we may assume that (9.2) holds in the sector $\tau_j < \arg z < \tau_{j+1}$, if and only if $n < m$. Use this to conclude than then the $V_{j+1}, \ldots, V_{j+j_0/2}$ are lower, the $V_{j+j_0/2+1}, \ldots, V_{j+j_0}$ upper triangular, and use the previous exercises. Finally, use (9.4) to compute the remaining matrices.

Section 9.3:

1. Observe $z\,Y'(z) = B(z)\,Y(z)$, and use partial integration of (9.6).
2. Estimate analogously to the proof of Theorem 23 (p. 80).

Section 9.4:

2. For $z_0 \in S_{j*(k)}$, observe the previous exercise to invert the integral representation of the auxiliary functions. Then, deform the path of integration in Borel's transform. Finally, check that the integral does not depend upon the choice of z_0.

Section 9.5:

2. Using Exercise 2 on p. 58, one can compute $\Phi_1(u;s;k)$, which is a 2-vector, with second component $c\,(\lambda_1-\lambda_2)^{-1}(u+\lambda_2)^{1-a-s}\,F(1+\alpha, 1+\beta; 2-a-s; z)/\Gamma(2-a-s)$, with $z=(u+\lambda_1)/(\lambda_1-\lambda_2)$. Using Exercise 6 on p. 27, one can then compute $C_{21}^{(k)}$. Assuming $2 \prec 1$, and choosing $\arg(\lambda_1-\lambda_2)$ in dependence of k accordingly, one finds

$$C_{21}^{(k)} = \frac{2\pi i\,(\lambda_1-\lambda_2)^{d-a}}{\Gamma(1+\alpha)\,\Gamma(1+\beta)}\,c.$$

Section 9.6:

2. Verify that the spectrum of A_0 is directly related to the data pairs of HLFFS.

Section 9.7:

2. Show $0 \le \prod_{m=n}^{n+p}(1+x_m) - 1 \le \exp[\sum_{m=n}^{n+p} x_m] - 1$, for $n, p \in \mathbb{N}$.
3. Proceed by induction with respect to m.
4. For $n, p \in \mathbb{N}$, observe

   $$T_{n+p}(z) - T_n(z) = [I + T_n(z)]\,\{[I + U_{n+p}(z) \cdot \ldots \cdot [I + U_{n+1}(z)] - I\},$$

   and then estimate as in the previous exercises.

Section 10.1:

1. Use the definition and justify interchanging the order of integration.

2. Let the jth integral be performed along $\arg z = d_j$; then we may choose d_{j-1} so that $2|d_{j-1} - d_j| \le \kappa_j^{-1}$.

Section 10.2:

4. Follows from the definition and (5.18) (p. 90).

6. Consider the operator $T = T_1 * \ldots * T_q$, and use Exercise 2 on p. 161.

Section 10.3:

2. Let $\tilde{T}_j = T_1 * \ldots * T_j$ and use the previous exercise, with a suitable choice of projections.

Section 10.4:

1. Apply Theorem 50 to $\hat{f}(z^p)$.

2. Note that $h_2 = \mathcal{S}\,(\hat{\mathcal{B}}_{k_2} \hat{f}_2)$ is holomorphic and of exponential growth at most k_2 in some $S(d_2, \varepsilon)$. Therefore, $g = g_1 + g_2$, with g_1 holomorphic at the origin, and $g_2 = \mathcal{L}_\kappa$ holomorphic in $S(d_2, \varepsilon + \pi/\kappa)$. This implies $\psi(z) = g_2(z) - g_2(z e^{2\pi i})$, hence ψ is holomorphic in $S(d_2 - \pi, \varepsilon + \pi(1/\kappa - 2))$.

3. Using Exercise 1 on p. 99, show the existence of $\psi(z)$, holomorphic in $S = S(d_2 - \pi, \varepsilon + \pi(1/\kappa - 2), r)$, having no holomorphic continuation beyond $|z| = r$, and so that $\psi(z) \cong_{1/\kappa} \hat{0}$ in S. Define $\hat{f}(z) = \sum f_n z^n$, $f_n = \Gamma(1 + n/k_1) \int_0^a \psi(w) w^{-n-1} dw$, $n \ge 1$, for some $a \in S$, and use the previous exercise.

4. Consider convergent series $\hat{g}_j$ with $\sum_{j=1}^q \hat{g}_j = \hat{0}$.

Section 10.5:

2. Use Theorem 50 and Exercise 5 on p. 73.

Section 10.6:

1. Argue as in the proof of Theorem 37 (p. 106).

2. Use Proposition 22 (p. 168) and Proposition 13 (p. 105).

Section 10.7:

1. Note $f_n = \Gamma(1 + n/2)\, g_n$, and show $\sum g_n\, z^n \in \mathbb{C}\{z\}_1$. The singular multidirections $d = (d_1, d_2)$ are those with $d_2 = 0$ modulo 2π. Observe that, owing to our identifying certain singular directions, these are indeed finitely many.

3. For the series in Exercise 1, show $h^{\pm}(z) = \int_0^{1/2} \psi(w)\,(w - z)^{-1}\, dw$, for $0 < \pm \arg z < \varepsilon$, so their difference equals $\psi(z)$.

Section 10.8:

1. Show that $d = (d_1, \ldots, d_q)$ is nonsingular for $\hat{f}$ if and only if $\tilde{d} = (d_1 - 2\pi/p, \ldots, d_q - 2\pi/p)$ is so for $\hat{g}$.

2. Show $p\, \hat{f}_j(z^p) = z^{-j} \sum_{\ell=0}^{p-1} \varepsilon^{-j\ell}\, \hat{f}(\varepsilon^{\ell} z)$, and use the previous exercise, together with Theorem 51 (p. 166).

3. Let k be the optimal type of summability for $\hat{f}$. If the corresponding values κ_j are all larger than $1/2$, apply Theorem 50 (p. 164). Otherwise, introduce additional k_j's so that then the theorem can be applied.

Section 11.1:

2. Substituting $t = x^{1/\alpha}$ in the previous exercise, show the representation

$$D_\alpha(z) = \alpha \int_0^{\infty(d)} t^{\alpha-1} \mathrm{e}^{-t^\alpha} e^{zt} dt, \quad -\pi/(2\alpha) < d < \pi/(2\alpha).$$

Section 11.3:

1. Show $\int_0^1 h(z\, x^s)\, dx = \int_0^1 f(z\,(1 - x)^s)\, g(z\, x^s)\, dx$ for $h = f *_k g$ and $s = 1/k$, and let $s \to 0$.

Section 12.1:

3. Show $k_m(z) = (1 - z\, \mathrm{e}^{am\,(z-1)})/(1 - z)$.

Section 12.4:

1. Determine $P(z)$ and $\tilde{E}(z)$ as follows: Set $\tilde{e}_{1j}(z) = e_{1j}(z)$, $1 \le j \le \nu$. Then not all $\tilde{e}_{1j}(z)$ can vanish at the origin, so assume $\tilde{e}_{11}(0) \ne 0$; otherwise, permute the columns of $\tilde{E}(z)$ accordingly. Now, assume that the entries of $P(z)$ and $\tilde{E}(z)$ in rows with numbers $\le j-1$ are known, and that $\tilde{e}_{\mu\mu}(0) \ne 0$, $1 \le \mu \le j-1$ (which is correct for $j = 2$). Then for $1 \le \ell < j$, we can recursively determine $p_{j\ell}(z)$ by requiring that $e_{j\ell}(z) - \sum_{\mu=1}^{\ell} p_{j\mu}(z)\, \tilde{e}_{\mu\ell}(z) = O(z^{k_\ell - k_j + 1})$; because of $\tilde{e}_{\ell\ell}(0) \ne 0$, this determines $p_{j\ell}(z)$ uniquely. Setting
$$\tilde{e}_{j\ell}(z) = e_{j\ell}(z) - \sum_{\mu=1}^{j-1} p_{j\mu}(z)\, \tilde{e}_{\mu\ell}(z),$$
$1 \le \ell \le \nu$, the first $j-1$ entries vanish as required, while of the remaining ones at least one cannot vanish at the origin. By a permutation of the columns with numbers $\ge j$ we then can arrange that $\tilde{e}_{jj}(0) \ne 0$.

2. Factor $E(z)$ as in the previous exercise, and observe that $z^K\, P(z)\, z^{-K}$ is an analytic transformation, while $z^K\, \tilde{E}(z)\, z^{-K}$ is entire.

Section A.1:

1. $x_{11} = x_{22} = 0$, $x_{12} = x_{21} = 1/2$.

2. Observe $J_2 = 0$, resp. $J_1 = 0$, in case $j = 1$, resp. $k = 1$.

3. Show first that λ is an eigenvalue of $\tilde{A}$ if and only if a nonzero $X \in \mathbb{C}^{\,k\times j}$ exists satisfying $A\,X - X\,B = \lambda X$. Then, use Lemma 25 to show that λ being an eigenvalue of $\tilde{A}$ implies $\lambda = \lambda_1 - \lambda_2$, with λ_1 being an eigenvalue of A and λ_2 one of B. Finally, consider matrices X of the form $X = x \cdot \tilde{x}^T$ to conclude that every such λ is indeed an eigenvalue of $\tilde{A}$.

Section A.2:

1. If A_{11} or A_{22} are not invertible, find a suitable vector $c \ne 0$ with $A\,c = 0$, if they are, verify the given formula for A^{-1}.

Section A.3:

1. Consider the power series expansion of ϕ about z_0 to show $|\phi(z)| \le c\,|z - z_0|^2$ for $|z - z_0| \le \varepsilon$, with sufficiently small $\varepsilon > 0$ and sufficiently large $c \ge 0$.

2. Show $|\phi(z)| \le (1 - 1/n)\,|z - z_0| + c\,|z - z_0|^2$, with ε and c as above.

Section B.1:

1. Use linearity of ϕ to show
$$\phi[(z-z_0)^{-1}(f(z)-f(z_0))] = (z-z_0)^{-1}[\phi(f(z))-\phi(f(z_0))],$$
and then let $z \to z_0$, using continuity of ϕ.

2. Observe
$$\frac{T(z)\,f(z)-T(z_0)\,f(z_0)}{z-z_0} = \frac{T(z)[f(z)-f(z_0)]}{z-z_0} - \frac{[T(z)-T(z_0)]f(z_0)}{z-z_0},$$
and analogously for α in place of T.

Section B.2:

1. Assume $f(z) = \sum_m^\infty f_n\,(z-z_0)^n$, for some $m \geq 0$, and set $g(z) = (z-z_0)^{-m} f(z)$. Then g is holomorphic in $D(z_0,\rho)$, and $g(z_k) = 0$. Letting $k \to \infty$ implies $g(0) = f_m = 0$.

2. Use Cauchy's integral theorem and proceed as in case of $\mathbb{E} = \mathbb{C}$.

5. Use Theorem 27 to show that all Taylor coefficients f_n, for $n \geq 1$, are equal to zero.

Section B.3:

1. Use holomorphic continuation by successive re-expansion.

2. Use the previous exercise.

3. Use the previous exercises together with Exercise 1 on p. 223.

4. For (a), use (B.8) and a change of variable $t = x\,u$. For (d), observe that we have shown the left-hand side of (B.12) to have a limit; to evaluate it, it suffices to consider positive real values of z.

5. Show $(1+a/z)^{a+z} = \exp[(a+z)\log(1+a/z)] \to \mathrm{e}^a$ for $|z| \to \infty$ in sectors $|\arg z| \leq \pi - \varepsilon$.

6. Observe that the left-hand side is $\Gamma(1+z)\,\Gamma(1+n)\,n^z/\Gamma(1+n+z)$ and use the previous exercise.

10. Justify interchanging the order of integration and use Cauchy's formula to evaluate the integral. Then, prove (B.14) for $\operatorname{Re} z > 0$, and use Exercises 3 and 7 to show the same for general $z \in \mathbb{C}$.

11. Use the identity theorem of Exercise 3 to see that the formula (B.14), proven for $\operatorname{Re} z < -1$, extends automatically.

Section B.4:

1. Under the assumptions made, there is exactly one w on the path γ for which $w^\alpha = z$.

2. The path γ may be replaced by one, for which the two radial parts, instead of the negative real axis, are along the rays $\arg w = \pm(\varepsilon + \pi/2)$, for arbitrarily small $\varepsilon > 0$. Doing so, the denominator in the integral never vanishes.

3. Use Theorem 69 (p. 233), together with Stirling's formula in Theorem 68 (p. 229).

Section B.5:

1. Study the function $f(z) = \exp[a\, z^k]$, $a \in \mathbb{C}$.

2. Either f is constant or of order larger than $\pi/(\beta - \alpha)$.

Section C.1:

1. Use Lemma 24 (p. 212) to show the existence of T so that
$$A = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -T & I \end{bmatrix},$$
and then use the rules for the exponential of a matrix.

2. If N is a $\nu \times \nu$ Jordan block, then
$$z^N = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ \log z & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \log^{\nu-1} z & \log^{\nu-2} z & \ldots & \log z & 1 \end{bmatrix}.$$

3. Verify that it suffices to consider A in Jordan canonical form, and then use the rules for computing exponentials of diagonal, resp. nilpotent, matrices.

4. Comparing the quadratic terms in the power series expansions of both sides, one finds as a necessary condition that $(A+B)^2 = A^2 + 2\,A\,B + B^2$, which holds if and only if A and B commute.

5. By induction, show
$$\frac{d}{dz}\,[A(z)]^n = n\,[A(z)]^{n-1}\,A'(z) = n\,A'(z)\,[A(z)]^{n-1}, \quad n \ge 1,$$
and then justify termwise differentiation of $\sum [A(z)]^n/n!$

Section C.2:

1. $A = T^{-1}\mathrm{diag}\,[0, 2\pi i]\,T$, hence $\mathrm{e}^A = T^{-1}\mathrm{diag}\,[\mathrm{e}^0, \mathrm{e}^{2\pi i}]\,T = I$.

2. By definition, branches of the matrix logarithm cannot have eigenvalues that differ by integer multiples of $2\pi i$.

3. Insert into (C.1) and justify termwise integration.

References

[1] L. V. Ahlfors, *Complex Analysis*, McGraw–Hill, New York, 1966.

[2] D. V. Anosov and A. A. Bolibruch, *The Riemann–Hilbert Problem*, Vieweg, 1994.

[3] D. G. Babbitt and V. S. Varadarajan, *Formal reduction of meromorphic differential equations: a group theoretic view*, Pac. J. of Math., 109 (1983), pp. 1–80.

[4] ———, *Local isoformal deformation theory of meromorphic differential equations near an irregular singularity*, Math. and Physical Sc., 247 (1988), pp. 583–700.

[5] ———, *Local moduli for meromorphic differential equations*, Astérisque, 169–170 (1989), pp. 1–217.

[6] I. Bakken, *On the central connection problem for a class of ordinary differential equations I*, Funkcialaj Ekvacioj, 20 (1977), pp. 115–127.

[7] ———, *On the central connection problem for a class of ordinary differential equations II*, Funkcialaj Ekvacioj, 20 (1977), pp. 129–156.

[8] W. Balser, *Einige Beiträge zur Invariantentheorie meromorpher Differentialgleichungen*, Habilitationsschrift, Universität Ulm, 1978.

[9] ———, *Growth estimates for the coefficients of generalized formal solutions, and representation of solutions using Laplace integrals and factorial series*, Hiroshima Math. J., 12 (1982), pp. 11–42.

[10] ———, *Solutions of first level of meromorphic differential equations*, Proc. Edinburgh Math. Soc., 25 (1982), pp. 183–207.

[11] ———, *Convergent power series expansions for the Birkhoff invariants of meromorphic differential equations; Part I, Definition of the coefficient functions*, Yokohama Math. J., 32 (1984), pp. 15–29.

[12] ———, *Convergent power series expansions for the Birkhoff invariants of meromorphic differential equations; Part II, A closer study of the coefficients*, Yokohama Math. J., 33 (1985), pp. 5–19.

[13] ———, *Explicit evaluation of the Stokes multipliers and central connection coefficients for certain systems of linear differential equations*, Math. Nachr., 138 (1988), pp. 131–144.

[14] ———, *Meromorphic transformation to Birkhoff standard form in dimension three*, J. Fac. Sc. Tokyo, 36 (1989), pp. 233–246.

[15] ———, *Analytic transformation to Birkhoff standard form in dimension three*, Funkcialaj Ekvacioj, 33 (1990), pp. 59–67.

[16] ———, *Dependence of differential equations upon parameters in their Stokes multipliers*, Pac. J. of Math., 149 (1991), pp. 211–229.

[17] ———, *A different characterization of multisummable power series*, Analysis, 12 (1992), pp. 57–65.

[18] ———, *Summation of formal power series through iterated Laplace integrals*, Math. Scandinavica, 70 (1992), pp. 161–171.

[19] ———, *Addendum to my paper: A different characterization of multisummable power series*, Analysis, 13 (1993), pp. 317–319.

[20] ———, *Calculation of the Stokes multipliers for a polynomial system of rank* 1 *having distinct eigenvalues at infinity*, Hiroshima Math. J., 23 (1993), pp. 223–230.

[21] ———, *From Divergent Power Series to Analytic Functions*, vol. 1582 of Lecture Notes in Math., Springer, 1994.

[22] ———, *Formal solutions of non-linear systems of ordinary differential equations*, in The Stokes Phenomenon and Hilbert's 16th Problem, B. Braaksma, G. Immink, and M. van der Put, eds., World Scientific, Singapore, 1995, pp. 25–49.

[23] ———, *An integral equation for normal solutions to meromorphic differential equations*, J. of Dynamical and Control Systems, 1 (1995), pp. 367–378.

[24] ———, *Existence and structure of complete formal solutions of non-linear meromorphic systems of ODE*, Asymptotic Analysis, 15 (1997), pp. 261–282.

[25] ———, *Moment methods and formal power series*, J. des Math. Pures et Appl., 76 (1997), pp. 289–305.

[26] ———, *Multisummability of complete formal solutions for non-linear systems of meromorphic ordinary differential equations*, Complex Variables, 34 (1997), pp. 19–24.

[27] ———, *Divergent solutions of the heat equation: on an article of Lutz, Miyake and Schäfke*, Pac. J. of Math., 188 (1999), pp. 53–63.

[28] ———, *Some remarks, examples, and questions concerning summability of formal factorial series*, Ulmer Seminare – Funktionalanalysis und Differentialgleichungen, University of Ulm, 1999.

[29] W. Balser and A. Beck, *Necessary and sufficient conditions for matrix summability methods to be stronger than multisummability*, Ann. Inst. Fourier Grenoble, 46 (1996), pp. 1349–1357.

[30] W. Balser and A. A. Bolibruch, *Transformation of reducible equations to Birkhoff standard form*, Ulmer Seminare – Funktionalanalysis und Differentialgleichungen, University of Ulm, 1997.

[31] W. Balser, B. L. J. Braaksma, J.-P. Ramis, and Y. Sibuya, *Multisummability of formal power series solutions of linear ordinary differential equations*, Asymptotic Analysis, 5 (1991), pp. 27–45.

[32] W. Balser and R. W. Braun, *Power series methods and multisummability*, Ulmer Seminare – Funktionalanalysis und Differentialgleichungen, University of Ulm, 1997. To appear in Math. Nachrichten.

[33] W. Balser, W. B. Jurkat, and D. A. Lutz, *Birkhoff invariants and Stokes multipliers for meromorphic linear differential equations*, J. of Math. Analysis and Appl., 71 (1979), pp. 48–94.

[34] ———, *A general theory of invariants for meromorphic differential equations; Part I, formal invariants*, Funkcialaj Ekvacioj, 22 (1979), pp. 197–221.

[35] ———, *A general theory of invariants for meromorphic differential equations; Part II, proper invariants*, Funkcialaj Ekvacioj, 22 (1979), pp. 257–283.

[36] ———, *A general theory of invariants for meromorphic differential equations; Part III, applications*, Houston J. of Math., 6 (1980), pp. 149–189.

[37] ——, *On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities; Part I*, SIAM J. of Math. Analysis, 12 (1981), pp. 691–721.

[38] ——, *Transfer of connection problems for meromorphic differential equations of rank $r \geq 2$ and representations of solutions*, J. of Math. Analysis and Appl., 85 (1982), pp. 488–542.

[39] ——, *Transfer of connection problems for first level solutions of meromorphic differential equations, and associated Laplace transforms*, J. reine und angew. Math., 344 (1983), pp. 149–170.

[40] ——, *Characterization of first level formal solutions by means of the growth of their coefficients*, J. of Differential Equ., 51 (1984), pp. 48–77.

[41] ——, *On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities; Part II*, SIAM J. of Math. Analysis, 19 (1988), pp. 398–443.

[42] W. Balser and M. Miyake, *Summability of formal solutions of certain partial differential equations*, Ulmer Seminare – Funktionalanalysis und Differentialgleichungen, University of Ulm, 1999. To appear in Acta. Sc. Math. Szeged.

[43] W. Balser and A. Tovbis, *Multisummability of iterated integrals*, Asymptotic Analysis, 7 (1993), pp. 121–127.

[44] M. A. Barkatou, *Rational Newton Algorithm for computing formal solutions of linear differential equations*, in Proceedings of ISSAC '88, Rome, Italy, ACM Press, 1988, pp. 183–195.

[45] ——, *An algorithm for computing a companion block diagonal form for a system of linear differential equations*, J. of App. Alg. in Eng. Comm. and Comp., 4 (1993), pp. 185–195.

[46] ——, *An algorithm to compute the exponential part of a formal fundamental matrix solution of a linear differential system*, J. App. Alg. in Eng. Comm. and Comp., 8 (1997), pp. 1–23.

[47] ——, *On rational solutions of systems of linear differential equations*, RR 973, IMAG Grenoble, 1997. To appear in J. of Symbolic Computation.

[48] M. A. Barkatou and A. Duval, *Sur la somme de certaines séries de factorielles*, Ann. Fac. Sc. Toulouse, 6 (1997), pp. 7–58.

[49] M. A. Barkatou and E. Pflügel, *An algorithm computing the regular formal solutions of a system of linear differential equations*, RR 988, LMC–IMAG, 1997. To appear in J. of Symbolic Computation.

[50] A. Beck, *Matrix-Summationsverfahren und Multisummierbarkeit*, Dissertation, Universität Ulm, 1995.

[51] D. Bertrand, *Travaux récents sur les points singuliers des équations différentielles linéaires*, in Sém. Bourbaki 1978/79, vol. 770 of Lecture Notes in Math., Springer, 1980, pp. 228–243.

[52] L. Bieberbach, *Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage dargestellt*, Springer, 1965.

[53] G. D. Birkhoff, *Singular points of ordinary linear differential equations*, Trans. of the Amer. Math. Soc., 10 (1909), pp. 436–470.

[54] ———, *Equivalent singular points of ordinary linear differential equations*, Math. Annalen, 74 (1913), pp. 134–139.

[55] ———, *The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations*, Proc. of the Amer. Acad. of Arts and Sc., 49 (1913), pp. 521–568.

[56] ———, *A theorem on matrices of analytic functions*, Math. Annalen, 74 (1913), pp. 122–133.

[57] ———, *Collected Mathematical Papers Vol. 1*, Dover Publications, New York, 1968.

[58] A. A. Bolibruch, *Construction of a Fuchsian equation from a monodromy representation*, Math. Notes of the Acad. of Sc. of USSR, 48 (1990), pp. 1090–1099.

[59] ———, *The Riemann-Hilbert problem*, Russian Math. Surveys, 45 (1990), pp. 1–47.

[60] ———, *Fuchsian systems with reducible monodromy and the Riemann-Hilbert problem*, in Global Analysis – Studies and Applications, Y. G. Borisovitch and Y. E. Gliklikh, eds., vol. 1520 of Lecture Notes in Mathematics, Springer, 1991, pp. 139–155.

[61] ———, *On analytic transformation to Birkhoff standard form*, Proc. of the Steklov Inst. of Math., 203 (1994), pp. 29–35.

[62] ———, *On analytic transformation to Birkhoff standard form*, Russian Acad. of Sc. Dokl. Math., 49 (1994), pp. 150–153.

[63] ——, *The Riemann-Hilbert problem and Fuchsian differential equations on the Riemann sphere*, in Proc. of the ICM Zürich 1994, Basel, 1995, Birkhäuser Verlag, pp. 1159–1168.

[64] ——, *On the Birkhoff standard form of linear systems of ODE*, Amer. Math. Soc. Translations, 174 (1996), pp. 169–179.

[65] B. L. J. Braaksma, *Inversion theorems for some generalized Fourier transforms, I*, Indag. Math., 28 (1966), pp. 275–299.

[66] ——, *Asymptotic analysis of a differential equation of Turrittin*, SIAM J. of Math. Analysis, 2 (1971), pp. 1–16.

[67] ——, *Erratum: Asymptotic analysis of a differential equation of Turrittin*, SIAM J. of Math. Analysis, 3 (1972), p. 175.

[68] ——, *Recessive solutions of linear differential equations with polynomial coefficients*, in Conference on the Theory of Ordinary and Partial Differential Equations, W. N. Everitt and B. D. Sleeman, eds., vol. 280 of Lecture Notes in Math., Springer, 1972, pp. 1–15.

[69] ——, *Multisummability and Stokes multipliers of linear meromorphic differential equations*, J. of Differential Equ., 92 (1991), pp. 45–75.

[70] ——, *Multisummability of formal power series solutions of nonlinear meromorphic differential equations*, Ann. Inst. Fourier Grenoble, 42 (1992), pp. 517–540.

[71] B. L. J. Braaksma and B. F. Faber, *Multisummability for some classes of difference equations*, Ann. Inst. Fourier Grenoble, 46 (1996), pp. 183–217.

[72] B. L. J. Braaksma and W. A. Harris, Jr., *Laplace integrals and factorial series in singular functional differential systems*, Applicable Analysis, 8 (1978), pp. 23–45.

[73] B. L. J. Braaksma and A. Schuitman, *Some classes of Watson transforms and related integral equations for generalized functions*, SIAM J. of Math. Analysis, 7 (1976), pp. 771–798.

[74] A. D. Brjuno, *Analytic form of differential equations*, Trans. Moscow Math. Soc., 25 (1971), pp. 131–288.

[75] N. G. d. Bruijn, *Asymptotic Methods in Analysis*, North-Holland Publ. Co., Amsterdam, 1958.

[76] M. Canalis-Durand, *Solution formelle Gevrey d'une équation différentielle singulièrement perturbée*, Asymptotic Analysis, 8 (1994), pp. 185–216.

[77] M. Canalis-Durand, J.-P. Ramis, R. Schäfke, and Y. Sibuya, *Gevrey solutions of singularly perturbed differential and difference equations*, tech. rep., IRMA Strasbourg, 1999. Accepted by J. reine und angew. Math.

[78] B. Candelbergher, J. C. Nosmas, and F. Pham, *Approche de la Résurgence*, Hermann, Paris, 1993.

[79] G. Chen, *An algorithm for computing the formal solutions of differential systems in the neighbourhood of an irregular singular point*, in Proceedings of ISSAC '90, 1990, pp. 231–235.

[80] ——, *Forme normale d'Arnold et réduction formelle des systèmes d'équations linéaires aux différences*, Aequat. Math., 54 (1997), pp. 264–288.

[81] G. Chen and A. Fahim, *Formal reduction of linear difference systems*, Pac. J. of Math., 182 (1998), pp. 37–54.

[82] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw–Hill, 1955.

[83] O. Costin, *On Borel summation and Stokes phenomena for rank-*1 *nonlinear systems of ordinary differential equations*, Duke Math. J., 93 (1998), pp. 289–344.

[84] P. Deligne, *Équations Différentielles à Points Singuliers Réguliers*, vol. 163 of Lecture Notes in Math., Springer, 1970.

[85] V. Dietrich, *Über eine notwendige und hinreichende Bedingung für regulär singuläres Verhalten von linearen Differentialgleichungssystemen*, Math. Zeitschr., 163 (1978), pp. 191–197.

[86] ——, *Zur Reduktion von linearen Differentialgleichungen*, Math. Annalen, 237 (1978), pp. 79–95.

[87] ——, *Über Reduzierbarkeit und maximale Ordnung bei linearen Differentialgleichungssystemen*, Complex Variables, 2 (1984), pp. 353–386.

[88] ——, *ELISE, an algorithm to compute asymptotic representations for solutions of linear differential equations, realized with the computer algebra system MAPLE*, J. Symbolic Comput., 14 (1992), pp. 85–92.

[89] R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, Academic Press, Oxford, 1973.

[90] T. M. Dunster and D. A. Lutz, *Convergent factorial series expansions for Bessel functions*, Proc. R. Soc. London Ser. A, 422 (1989), pp. 7–21.

[91] T. M. Dunster, D. A. Lutz, and R. Schäfke, *Convergent Liouville-Green expansions for second order linear differential equations, with an application to Bessel functions*, Proc. R. Soc. London Ser. A, 440 (1993), pp. 37–54.

[92] A. Duval, *Lemmes d'Hensel et factorisation formelle pour les opérateurs aux différences*, Funkcialaj Ekvacioj, 26 (1983), pp. 349–368.

[93] A. Duval and C. Mitschi, *Matrices de Stokes et groupe de Galois des équations hypergéometriques confluents generalisées*, Pac. J. Math., 138 (1989), pp. 25–56.

[94] J. Ecalle, *Les fonctions résurgentes I–II*, Publ. Math. d'Orsay, Université Paris Sud, 1981.

[95] ———, *Les fonctions résurgentes III*, Publ. Math. d'Orsay, Université Paris Sud, 1985.

[96] ———, *Introduction à l'Accélération et à ses Applications*, Travaux en Cours, Hermann, Paris, 1993.

[97] A. M. Emamzadeh, *Numerical investigations into the Stokes phenomenon I*, J. Inst. Math. Appl., 19 (1977), pp. 77–86.

[98] ———, *Numerical investigations into the Stokes phenomenon II*, J. Inst. Math. Appl., 19 (1977), pp. 149–157.

[99] ———, *A numerical method for the calculation of the Stokes constants*, Appl. Sci. Res., 34 (1978), pp. 161–178.

[100] A. Erdélyi, *Higher Transcendental Functions*, McGraw–Hill, 1953.

[101] ———, *Asymptotic Expansions*, Dover Publications, New York, 1956.

[102] B. F. Faber, *Summability theory for analytic difference and differential–difference equations*, Ph.D. thesis, Rijksuniversiteit Groningen, 1998.

[103] W. B. Ford, *Studies on Divergent Series and Summability & The Asymptotic Developments of Functions defined by MacLaurin Series*, Chelsea, 1960.

[104] A. Fruchard and R. Schäfke, *On the Borel transform*, C. R. Acad. Sci., 323 (1996), pp. 999–1004.

[105] F. R. Gantmacher, *Theory of Matrices, vol. I & II*, Chelsea, 1959.

[106] R. Gérard and A. H. M. Levelt, *Invariants mesurant l'irrégularité en un point singulier des systèmes d'équations différentielles linéaires*, Ann. Inst. Fourier Grenoble, 23 (1973), pp. 157–195.

[107] R. Gérard and D. A. Lutz, *Convergent factorial series solutions of singular operator equations*, Analysis, 10 (1990), pp. 99–145.

[108] R. Gérard and H. Tahara, *Formal power series solutions of nonlinear first order partial differential equations*, Funkcialaj Ekvacioj, 41 (1998), pp. 133–166.

[109] H. E. Gollwitzer and Y. Sibuya, *Stokes multipliers for subdominant solutions of second order differential equations with polynomial coefficients*, J. reine u. angew. Math., 243 (1970), pp. 98–119.

[110] I. J. Good, *Note on the summation of a classical divergent series*, J. London Math. Soc., 16 (1941), pp. 180–182.

[111] V. P. Gurarij and V. I. Matsaev, *Stokes multipliers for systems of linear ordinary differential equations of first order*, Soviet Math. Dokl., 31 (1985), pp. 52–56.

[112] G. H. Hardy, *On the summability of series by Borel's and Mittag-Leffler's methods*, J. London Math. Soc., 9 (1934), pp. 153–157.

[113] ———, *Note on a divergent series*, Proc. Cambr. Phil. Soc., 37 (1941), pp. 1–8.

[114] W. A. Harris Jr., *Characterization of linear differential systems with a regular singular point*, Proc. Edinburgh Math. Soc., 18 (1972), pp. 93–98.

[115] J. Heading, *The Stokes phenomenon and certain n-th order differential equations I, II*, Proc. Cambridge Phil. Soc., 53 (1957), pp. 399–441.

[116] ———, *The Stokes phenomenon and the Whittaker function*, J. London Math. Soc., 37 (1962), pp. 195–00.

[117] A. Hilali and A. Wazner, *Calcul des invariants de Malgrange et de Gérard et Levelt d'un système différentiel linéaire en un point singulier irrégulier*, J. Differential Equ., 69 (1987).

[118] ———, *Formes super–irréducibles des systèmes différentiels linéaires*, Numer. Math., 50 (1987), pp. 429–449.

[119] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen (Dritte Mitt.)*, Nachr. Ges. der Wiss., Göttingen, (1905), pp. 307–338.

[120] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Wiley, New York, 1976.

[121] F. L. Hinton, *Stokes multipliers for a class of ordinary differential equations*, J. Math. Phys. (10), 20 (1979), pp. 2036–2046.

[122] M. v. Hoeij, *Rational solutions of the mixed differential equation and its application to factorization of differential operators*, in Proceedings of ISSAC '96, ACM Press, 1996.

[123] ———, *Factorization of differential operators with rational function coefficients*, J. Symbolic Comput., 24 (1997), pp. 537–561.

[124] ———, *Formal solutions and factorization of differential operators with power series coefficients*, J. Symbolic Comput., 24 (1997), pp. 1–30.

[125] J. Horn, *Fakultätenreihen in der Theorie der linearen Differentialgleichungen*, Math. Annalen, 71 (1912), pp. 510–532.

[126] ———, *Integration linearer Differentialgleichungen durch Laplacesche Integrale und Fakultätenreihen*, Jahresber. DMV, 25 (1916), pp. 74–83.

[127] ———, *Integration linearer Differentialgleichungen durch Laplacesche Integrale I*, Math. Zeitschr., 49 (1944), pp. 339–350.

[128] ———, *Integration linearer Differentialgleichungen durch Laplacesche Integrale II*, Math. Zeitschr., 49 (1944), pp. 684–701.

[129] P. Hsieh, *On the asymptotic integration of* $x^2y'' - p(x)y = 0$, Arch. Math. Brno (2), 14 (1978), pp. 75–83.

[130] P. Hsieh and Y. Sibuya, *On the asymptotic integration of second order linear ordinary differential equations with polynomial coefficients*, J. Math. Anal. and Appl., 16 (1966), pp. 84–103.

[131] M. Hukuhara, *Fifty years of ordinary differential equations I*, Sûugaku, 34 (1982), pp. 164–171.

[132] ———, *Fifty years of ordinary differential equations II*, Sûugaku, 34 (1982), pp. 262–269.

[133] G. K. Immink, *Asymptotic of analytic difference equations*, Ph.D. thesis, Rijksuniversiteit Groningen, 1983.

[134] ——, *Asymptotics of Analytic Difference Equations*, vol. 1085 of Lecture Notes in Math., Springer, 1984.

[135] ——, *A note on the relationship between Stokes multipliers and formal solutions of analytic differential equations*, SIAM J. of Math. Anal., 21 (1990), pp. 782–792.

[136] ——, *On the asymptotic behavior of the coefficients of asymptotic power series and its relevance to Stokes phenomena*, SIAM J. of Math. Anal., 22 (1991), pp. 524–542.

[137] ——, *Multi-summability and the Stokes phenomenon*, J. Dynamical and Control Systems, 1 (1995), pp. 483–534.

[138] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1956.

[139] M. Iwano, *Intégration analytique d'un système d'équations non linéaires dans le voisinage d'un point singulier I*, Ann. Mat. Pura Appl., 44 (1957), pp. 261–292.

[140] ——, *Intégration analytique d'un système d'équations non linéaires dans le voisinage d'un point singulier II*, Ann. Mat. Pura Appl., 47 (1959), pp. 91–149.

[141] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida, *From Gauss to Painlevé: A Modern Theory of Special Functions*, Vieweg Verlag, Wiesbaden, 1991.

[142] F. Jung, F. Naegele, and J. Thomann, *An algorithm of multi-summation of formal power series solutions of linear ODE equations*, Math. and Computers in Simulation, 42 (1996), pp. 409–425.

[143] W. B. Jurkat, *Meromorphe Differentialgleichungen*, vol. 637 of Lecture Notes in Math., Springer, 1978.

[144] ——, *Summability of asymptotic power series*, Asymptotic Analysis, 7 (1993), pp. 239–250.

[145] W. B. Jurkat and D. A. Lutz, *On the order of solutions of analytic linear differential equations*, Proc. London Math. Soc., 22 (1971), pp. 465–482.

[146] W. B. Jurkat, D. A. Lutz, and A. Peyerimhoff, *Effective solutions for meromorphic second order differential equations*, in Symp. Ord. Differential Equ., vol. 312 of Lecture Notes in Math., Springer, 1973, pp. 100–107.

[147] ———, *Birkhoff invariants and effective calculations for meromorphic linear differential equations; Part I*, J. of Math. Analysis and Appl., 53 (1976), pp. 438–470.

[148] ———, *Birkhoff invariants and effective calculations for meromorphic linear differential equations; Part II*, Houston J. Math., 2 (1976), pp. 207–238.

[149] ———, *Invariants and canonical forms for meromorphic second order differential equations*, in Proc. 2nd Scheveningen Conference on Differential Equations, North-Holland Press, Amsterdam, 1976, pp. 181–187.

[150] N. Kazarinoff and R. McKelvey, *Asymptotic solutions of differential equations in a domain containing a regular singular point*, Canadian J. Math., 8 (1956), pp. 97–104.

[151] T. Kimura, *Analytic theory of ordinary differential equations IV*, in Global Theory of Nonlinear Differential Equations, Recent Progress of Natural Sciences in Japan, vol. 1, Science Council of Japan, Tokyo, 1976, pp. 47–55.

[152] H. W. Knobloch, *Zusammenhänge zwischen konvergenten und asymptotischen Entwicklungen bei Lösungen linearer Differentialgleichungs-Systeme vom Range 1*, Math. Annalen, 134 (1958), pp. 260–288.

[153] H. v. Koch, *Sur une application des déterminants infinis à la théorie des équations différentielles linéaires*, Acta Math., 15 (1891/92).

[154] M. Kohno, *The convergence condition of a series appearing in connection problems and the determination of Stokes' multipliers*, Publ. RIMS Kyoto Univ., 3 (1968), pp. 337–350.

[155] ———, *On the calculation of the approximate values of Stokes' multipliers*, Publ. RIMS Kyoto Univ., 4 (1968), pp. 277–298.

[156] ———, *A two point connection problem for n–th order single linear ordinary differential equations with an irregular singular point of rank two*, Japanese J. Math., 39 (1970), pp. 11–62.

[157] ———, *A two point connection problem for general linear ordinary differential equations*, Hiroshima Math. J., 4 (1974), pp. 293–338.

[158] ———, *A two point connection problem for n–th order single linear ordinary differential equations with an irregular singular point of rank two*, Japanese J. Math., 42 (1974), pp. 39–42.

[159] ——, *A two point connection problem*, Hiroshima Math. J., 9 (1979), pp. 61–135.

[160] ——, *Derivatives of Stokes multipliers*, Hiroshima Math. J., 14 (1984), pp. 247–256.

[161] M. Kohno and T. Yokoyama, *A central connection problem for a normal system of linear differential equations*, Hiroshima Math. J., 14 (1984), pp. 257–263.

[162] V. P. Kostov, *The Stokes multipliers and the Galois group of a non-Fuchsian system and the generalized Phragmen-Lindelöf principle*, Funkcialaj Ekvacioj, 36 (1993), pp. 329–357.

[163] M. A. Kovalevskij, *Construction of the Stokes multipliers for an equation with two singular points*, Vestnik Leningrad Univ. Math., 14 (1982), pp. 135–141.

[164] ——, *Determination of the connection between two fundamental families of solutions of a linear ordinary differential equation*, Vestnik Leningrad Univ. Math., 14 (1982), pp. 39–45.

[165] T. Kurth and D. Schmidt, *On the global representation of the solutions of second-order linear differential equations having an irregular singular point of rank one in ∞ by series in terms of confluent hypergeometric functions*, SIAM J. of Math. Analysis, 17 (1986), pp. 1086–1103.

[166] W. Lay and S. Y. Slavyanov, *The central two–point connection problem for the Heun class of ODEs*, J. Phys. A: Math. Gen., 31 (1998), pp. 4249–4261.

[167] A. H. M. Levelt, *Jordan decomposition for a class of singular differential operators*, Ark. Mat., 13 (1975), pp. 1–27.

[168] C.-H. Lin and Y. Sibuya, *Some applications of isomonodromic deformations to the study of Stokes multipliers*, J. Fac. Sci. Tokyo, 36 (1989), pp. 649–663.

[169] M. Loday-Richaud, *Introduction à la multisommabilité*, Gaz. Math. Soc. France, 44 (1990), pp. 41–63.

[170] ——, *Solutions formelles des systèmes différentiels linéaires méromorphes et sommation*, Expos. Math., 13 (1995), pp. 115–162.

[171] D. A. Lutz, *On systems of linear differential equations having regular singular solutions*, J. Differential Equ., 3 (1967), pp. 311–322.

[172] ——, *Some characterizations of systems of linear differential equations having regular singular solutions*, Trans. Amer. Math. Soc., 126 (1967), pp. 427–441.

[173] ——, *Asymptotic behavior of solutions of linear systems of ordinary differential equations near an irregular singular point*, Amer. J. Math., 91 (1969), pp. 95–105.

[174] ——, *On the reduction of rank of linear differential systems*, Pac. J. of Math., 42 (1972), pp. 153–164.

[175] ——, *Connection problems in the parameterless case: Progress and more problems*, in Sing. Pert. and Asympt., R. Meyer and S. Parter, eds., J. Wiley, New York, 1980, pp. 357–378.

[176] D. A. Lutz, M. Miyake, and R. Schäfke, *On the Borel summability of divergent solutions of the heat equation*, Nagoya Math. J., 154 (1999), pp. 1–29.

[177] D. A. Lutz and R. Schäfke, *On the identification and stability of formal invariants for singular differential equations*, Linear Algebra and Appl., 72 (1985), pp. 1–46.

[178] ——, *Calculating connection coefficients for meromorphic differential equations*, Complex Variables, 34 (1997), pp. 145–170.

[179] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, 1988.

[180] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for Special Functions of Physics*, Springer, 1966.

[181] H. Majima, *Asymptotic Analysis for Integrable Connections with Irregular Singular Points*, vol. 1075 of Lecture Notes in Math., Springer, 1984.

[182] B. Malgrange, *Sur les points singuliers des équations différentielles*, Enseign. Math., 20 (1974), pp. 147–176.

[183] ——, *Remarques sur les équations différentielles à points singuliers irréguliers*, in Équations Différentielles et Systèmes de Pfaff dans le Champ Complexe, vol. 712 of Lecture Notes in Math., Springer, 1979, pp. 77–86.

[184] ——, *Sommation de séries divergentes*, Expos. Math., 13 (1995), pp. 163–222.

[185] B. Malgrange and J.-P. Ramis, *Fonctions multisommables*, Ann. Inst. Fourier Grenoble, 42 (1991), pp. 1–16.

[186] J. Martinet and J.-P. Ramis, *Théorie de Galois différentielle et resommation*, in Computer Algebra and Differential Equations, E. Tournier, ed., Academic Press, New York, 1989.

[187] ——, *Elementary acceleration and multisummability*, Ann. Inst. Henri Poincaré, Physique Theorique, 54 (1991), pp. 331–401.

[188] P. Masani, *On a result of G. D. Birkhoff on linear differential systems*, Proc. Amer. Math. Soc., 10 (1959), pp. 696–698.

[189] J. A. M. McHugh, *A novel solution of a lateral connection problem*, J. Differential Equ., 13 (1973), pp. 374–383.

[190] M. Miyake, *Relations of equations of Euler, Hermite and Weber via the heat equation*, Funkcialaj Ekvacioj, 36 (1993), pp. 251–273.

[191] M. Miyake and Y. Hashimoto, *Newton polygons and Gevrey indices for linear partial differential operators*, Nagoya Math. J., 128 (1992), pp. 15–47.

[192] M. Miyake and M. Yoshino, *Fredholm property for differential operators on formal Gevrey space and Toeplitz operator method*, C. R. Acad. Bulgare de Sciences, 47 (1994), pp. 21–26.

[193] ——, *Wiener–Hopf equation and Fredholm property of the Goursat problem in Gevrey space*, Nagoya Math. J., 135 (1994), pp. 165–196.

[194] ——, *Toeplitz operators and an index theorem for differential operators on Gevrey spaces*, Funkcialaj Ekvacioj, 38 (1995), pp. 329–342.

[195] J. Moser, *The order of the singularity in Fuchs' theory*, Math. Zeitschr., 72 (1960), pp. 379–398.

[196] B. T. M. Murphy and A. D. Wood, *Hyperasymptotic solutions of second–order ordinary differential equations with a singularity of arbitrary integer rank*, Methods Appl. Analysis, 4 (1997), pp. 250–260.

[197] F. Naegele and J. Thomann, *Algorithmic approach of the multisummation of formal power series solutions of linear ODE applied to the Stokes Phenomena*, in The Stokes Phenomenon and Hilbert's 19th Problem, B. Braaksma, G. Immink, and M. van der Put, eds., World Scientific, Singapore, 1995, pp. 197–213.

[198] F. Naundorf, *Globale Lösungen von gewöhnlichen linearen Differentialgleichungen mit zwei stark singulären Stellen*, Dissertation, Universität Heidelberg, 1974.

[199] ——, *A connection problem for second order linear differential equations with irregular singular points*, SIAM J. Math. Analysis, 7 (1976), pp. 157–175.

[200] ——, *Ein Verfahren zur Lösung des Zusammenhangproblems bei linearen Differentialgleichungen zweiter Ordnung mit mehreren singulären Stellen*, Zeitschr. Angew. Math. Mech., 59 (1979), pp. 273–275.

[201] F. Nevanlinna, *Zur Theorie der asymptotischen Potenzreihen*, Ann. Acad. Sci. Fenn. Ser. A 1 Math. Dissertationes, 12 (1918), pp. 1–81.

[202] H. E. Newell Jr., *The asymptotic forms of the solution of an ordinary linear matrix equation in the complex domain*, Duke Math. J., 9 (1942), pp. 245–258.

[203] D. J. Newman, *An entire function bounded in every direction*, Amer. Math. Monthly, 83 (1976), pp. 192–193.

[204] N. E. Nörlund, *Leçons sur les séries l'interpolation*, Gauthier–Villars, 1926.

[205] K. Okubo, *A global representation of a fundamental set of solutions and a Stokes phenomenon for a system of linear ordinary differential equations*, J. Math. Soc. Japan, 15 (1963), pp. 268–288.

[206] ——, *Connection problem for systems of linear differential equations*, in Japan-United States Seminar on Ordinary Diff. and Functional Eqs., vol. 243 of Lecture Notes in Math., Springer, 1971, pp. 238–248.

[207] K. Okubo and K. Takano, *Generalized hypergeometric functions.* in: K. Okubo, On the Group of Fuchsian Equations, Progress Report, The Ministry of Education, Science and Culture, Japan, 1981.

[208] K. Okubo, K. Takano, and S. Yoshida, *A connection problem for the generalized hypergeometric equation*, Funkcialaj Ekvacioj, 31 (1988), pp. 483–495.

[209] A. B. Olde Daalhuis, *Hyperterminants I*, J. Comput. Appl. Math., 76 (1996), pp. 255–264.

[210] A. B. Olde Daalhuis and F. W. J. Olver, *Hyperasymptotic solutions of second–order linear differential equations I*, Methods Appl. Analysis, 2 (1995), pp. 173–197.

[211] ——, *On the calculation of Stokes' multipliers for linear differential equations of the second order*, Methods Appl. Analysis, 2 (1995), pp. 348–367.

[212] F. W. J. Olver, *Introduction to Asymptotics and Special Functions*, Academic Press, 1974.

[213] S. Ouchi, *Formal solutions with Gevrey type estimates of nonlinear partial differential equations*, J. Math. Sc. Univ. Tokyo, 1 (1994), pp. 205–237.

[214] ——, *Singular solutions with asymptotic expansion of linear partial differential equations in the complex domain*, Publ. RIMS Kyoto University, 34 (1998), pp. 291–311.

[215] R. B. Paris, *On the asymptotic expansions of solutions of an nth order linear differential equation*, Proc. Roy. Soc. Edinburgh, 85 A (1980), pp. 15–57.

[216] R. B. Paris and A. D. Wood, *The asymptotic expansion of solutions of the differential equation* $u^{iv} + \lambda^2[(z^2+c)u'' + azu' + bu] = 0$ *for large* $|z|$, Phil. Trans. Roy. Soc. London, 293 (1979), pp. 511–533.

[217] ——, *On the asymptotic expansion of solutions of an nth order linear differential equation with power coefficients*, Proc. Roy. Irish Acad., 85 A (1985), pp. 201–220.

[218] ——, *Asymptotics of High Order Differential Equations*, vol. 129 of Pitman Research Notes in Math. Series, Longman, Harlow, U.K., 1986.

[219] F. Pittnauer, *Vorlesungen über asymptotische Reihen*, vol. 301 of Lecture Notes in Math., Springer, 1972.

[220] J. Plemelj, *Riemannsche Funktionenscharen mit gegebener Monodromiegruppe*, Monatsh. f. Math. u. Phys., 19 (1908), pp. 211–246.

[221] ——, *Problems in the Sense of Riemann and Klein*, Interscience Publications, New York, 1964.

[222] H. Poincaré, *Sur les intégrales irregulières des équations linéaires*, Acta Math., 8 (1886), pp. 295–344.

[223] C. Praagman, *The formal classification of linear difference operators*, Proc. Kon. Ned. Ac. Wet., 86 (1983), pp. 249–261.

[224] M. v. d. Put and M. F. Singer, *Galois Theory of Difference Equations*, vol. 1666 of Lecture Notes in Math., Springer, 1997.

[225] J.-P. Ramis, *Dévissage Gevrey*, Astérisque, 59-60 (1978), pp. 173–204.

[226] ——, *Les séries k-sommable et leurs applications*, in Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory, D. Iagolnitzer, ed., vol. 126 of Lecture Notes in Physics, Springer, 1980, pp. 178–199.

[227] ——, *Phénomène de Stokes et filtration Gevrey sur le groupe de Picard-Vessiot*, C. R. Acad. Sci., 301 (1985), pp. 165–167.

[228] ——, *Phénomène de Stokes et resommation*, C. R. Acad. Sci., 301 (1985), pp. 99–102.

[229] ——, *Séries Divergentes et Théories Asymptotiques*, vol. 121 of Panoramas et synthèses, Soc. Math. France, Paris, 1993.

[230] J.-P. Ramis and Y. Sibuya, *A new proof of multisummability of formal solutions of non linear meromorphic differential equations*, Annal. Inst. Fourier Grenoble, 44 (1994), pp. 811–848.

[231] V. Reuter, *Verbindungsprobleme bei meromorphen Differentialgleichungen vom Poincaré-Rang* $r \geq 2$, Dissertation, Universität Ulm, 1991.

[232] ——, *On connection problems for differential equations of arbitrary Poincaré rank*, C. R. Acad. Sci., 315 (1992), pp. 1371–1374.

[233] J. F. Ritt, *On the derivatives of a function at a point*, Ann. Math., 18 (1916), pp. 18–23.

[234] A. Ronveaux, ed., *Heun's Differential Equations*, Oxford Science Publisher, New York, 1995.

[235] F. W. Schäfke, *Einführung in die Theorie der speziellen Funktionen der mathematischen Physik*, Springer, 1963.

[236] F. W. Schäfke and D. Schmidt, *Gewöhnliche Differentialgleichungen, Die Grundlagen der Theorie im Reellen und Komplexen*, vol. 108 of Heidelberger Taschenbücher, Springer, 1973.

[237] R. Schäfke, *Über das globale analytische Verhalten der Lösungen der über die Laplace–Transformation zusammenhängenden Differentialgleichungen* $t\,x'(t) = (A + tB)\,x$ *und* $(s - B)\,v' = (\rho - A)\,v$, Dissertation, Essen, 1979.

[238] ——, *The connection problem for two neighboring regular singular points of general linear complex ordinary differential equations*, SIAM J. Math. Anal., 11 (1980), pp. 863–875.

[239] ——, *A connection problem for a regular and an irregular singular point of complex ordinary differential equations*, SIAM J. Math. Anal., 15 (1984), pp. 253–271.

[240] ——, *Über das globale Verhalten der Normallösungen von $x'(t) = (B+t^{-1}A)\,x(t)$, und zweier Arten von assoziierten Funktionen*, Math. Nachr., 121 (1985), pp. 123–145.

[241] R. Schäfke and D. Schmidt, *The connection problem for general linear ordinary differential equations at two regular singular points with applications in the theory of special functions*, SIAM J. Math. Anal., 11 (1980), pp. 848–862.

[242] R. Schäfke and H. Volkmer, *On the reduction of the Poincaré rank of singular systems of ordinary differential equations*, J. reine u. angew. Math., 365 (1986), pp. 80–96.

[243] S. Schlosser-Haupt and H. Wyrwich, *Über die Stokesschen Multiplikatoren gewisser linearer Differentialgleichungen n-ter Ordnung*, Math. Nachr., 95 (1987), pp. 265–275.

[244] D. Schmidt, *Die Lösung der linearen Differentialgleichung 2. Ordnung um zwei einfache Singularitäten durch Reihen nach hypergeometrischen Funktionen*, J. reine u. angew. Math., 309 (1979), pp. 127–148.

[245] ——, *Global representations for the solutions of second-order meromorphic differential equations by special functions*, in Ordinary and partial differential equations, Vol. III, Pitman Research Notes in Mathematics, Vol. 254, B. D. Sleeman and R. J. Jarvis, eds., Longman, Harlow, U.K., 1991, pp. 183–207.

[246] A. Schuitman, *A class of integral transforms and associated function spaces*, Ph.D. thesis, TH Delft, 1985.

[247] Y. Sibuya, *Stokes multipliers of subdominant solutions of the differential equation $y'' - (x^3 + \lambda)y = 0$*, Proc. Amer. Math. Soc., 18 (1967), pp. 238–243.

[248] ——, *Subdominant solutions of the differential equation $y'' - \lambda^2(x - a_1)\,(x - a_2)\ldots(x - a_m)y = 0$*, Acta Math., 119 (1967), pp. 235–273.

[249] ——, *Subdominant solutions admitting a prescribed Stokes phenomenon*, in Int. Conf. on Differential Equations, Academic Press, New York, 1975, pp. 709–738.

[250] ——, *Stokes' phenomena*, Bull. Amer. Math. Soc., 83 (1977), pp. 1075–1077.

[251] ——, *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, vol. 82 of Transl. Math. Monographs, Amer. Math. Soc., Providence, R.I., 1990.

[252] ———, *Gevrey asymptotics and Stokes multipliers*, in Differential Equ. and Comp. Algebra, M. F. Singer, ed., Academic Press, 1991, pp. 131–147.

[253] Y. Sibuya and S. Sperber, *Some new results on power series solutions of algebraic differential equations*, in Proc. of Advanced Sem. on Singular Perturbations and Asymptotics, Academic Press, New York, 1980, pp. 379–404.

[254] Y. Sibuya and T. Tabara, *Calculation of a Stokes multiplier*, Asympt. Analysis, 13 (1996), pp. 95–107.

[255] S. Y. Slavyanov, *On the question of the Stokes phenomenon for the equation* $y''(z) - z^m y(z) = 0$, Sov. Phys. Dokl., 30 (1985).

[256] V. R. Smilyansky, *Stokes multipliers for systems of linear ordinary differential equations I*, Differential Equations, 6 (1970), pp. 375–384.

[257] B. Sternin and V. Shatalov, *Borel-Laplace Transform and Asymptotic Theory*, CRC–Press, London, 1995.

[258] G. G. Stokes, *On the discontinuity of arbitrary constants which appear in divergent developments*, Trans. Camb. Phil. Soc., 10 (1857), pp. 106–128.

[259] T. J. Tabara, *A locally prescribed Stokes phenomenon*, Funkcialaj Ekvacioj, 35 (1992), pp. 429–450.

[260] J. Thomann, *Resommation de séries formelles*, Numer. Math., 58 (1990), pp. 503–535.

[261] ———, *Resommation de séries formelles solutions d'équations différentielles linéaires ordinaires du second ordre dans le champ complexe au voisinage de singularités irrégulières*, Numer. Math., 58 (1990), pp. 502–535.

[262] ———, *Procédés formels et numériques de sommation de séries solutions d'équations différentielles*, Expos. Math., 13 (1995), pp. 223–246.

[263] E. Tournier, Solutions formelles d'équations différentielles. Thèse d'Etat, Université de Grenoble, 1987.

[264] A. Tovbis, *On a method of constructing Stokes multipliers*, Sov. Math. Dokl., 35 (1987), pp. 202–206.

[265] ———, *Lateral connection problem and Stokes phenomenon for certain functional spaces*, Asympt. Anal., 4 (1991), pp. 215–233.

[266] W. J. Trjitzinsky, *Laplace integrals and factorial series in the theory of linear differential and linear difference equations*, Trans. Amer. Math. Soc., 37 (1935), pp. 80–146.

[267] H. L. Turrittin, *Stokes multipliers for asymptotic solutions of a certain differential equation*, Trans. Amer. Math. Soc., 68 (1950), pp. 304–329.

[268] ———, *Convergent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point*, Acta Math., 93 (1955), pp. 27–66.

[269] ———, *The formal theory of irregular homogeneous linear difference and differential equations*, Bol. Soc. Mat. Mexicana, (1960), pp. 225–264.

[270] ———, *Reducing the rank of ordinary differential equations*, Duke Math. J., 30 (1963), pp. 271–274.

[271] ———, *Reduction of ordinary differential equations to the Birkhoff canonical form*, Trans. Amer. Math. Soc., 107 (1963), pp. 485–507.

[272] ———, *Stokes multipliers for the differential equation* $y^{(n)}(x) - y(x)/x = 0$, Funkcialaj Ekvacioj, 9 (1966), pp. 261–272.

[273] J. Vandamme, *Problème de Riemann–Hilbert pour une représentation de monodromie triangulaire supérieure*, Ph.D. thesis, Université de Nice, 1998.

[274] V. S. Varadarajan, *Meromorphic differential equations*, Expos. Math., 9 (1991), pp. 97–188.

[275] ———, *Linear meromorphic differential equations: a modern point of view*, Bull. Amer. Math. Soc., 33 (1996), pp. 1–42.

[276] E. Wagenführer, *Über regulär singuläre Lösungen von Systemen linearer Differentialgleichungen; I*, J. reine u. angew. Math., 267 (1974), pp. 90–114.

[277] ———, *Über regulär singuläre Lösungen von Systemen linearer Differentialgleichungen; II*, J. reine u. angew. Math., 272 (1975), pp. 150–175.

[278] G. Wallet, *Surstabilité pour une équation différentielle analytique en dimension* 1, Ann. Inst. Fourier Grenoble, 40 (1990), pp. 557–595.

[279] ———, *Singularité analytique et perturbation singulière en dimension* 2, Bullet. Soc. Math. France, 122 (1994), pp. 185–208.

[280] W. Wasow, *Connection problems for asymptotic series*, Bull. Amer. Math. Soc., 74 (1968), pp. 831–853.

[281] ——, *Asymptotic Expansions of Ordinary Differential Equations*, Dover Publications, New York, 1987.

[282] G. N. Watson, *A theory of asymptotic series*, Trans. Royal Soc. London, Ser. A, 211 (1911), pp. 279–313.

[283] ——, *The transformation of an asymptotic series into a convergent series of inverse factorials*, Rend. Circ. Palermo, 34 (1912), pp. 41–88.

[284] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, N.Y., 1941.

[285] H. Wyrwich, *Eine explizite Lösung des "central connection problem" für eine gewöhnliche lineare Differentialgleichung n-ter Ordnung mit Polynomkoeffizienten*, Dissertation, Universität Dortmund, 1974.

[286] ——, *An explicit solution of the central connection problem for an nth order linear differential equation with polynomial coefficients*, SIAM J. Math. Anal., 8 (1977), pp. 412–422.

[287] T. Yokoyama, *On connection formulas for a fourth order hypergeometric system*, Hiroshima Math. J., 15 (1985), pp. 297–320.

[288] ——, *Characterization of connection coefficients for hypergeometric systems*, Hiroshima Math. J., 17 (1987), pp. 219–233.

[289] ——, *On the structure of connection coefficients for hypergeometric systems*, Hiroshima Math. J., 18 (1988), pp. 309–339.

[290] M. Yoshida, *Fuchsian Differential Equations*, Vieweg, 1987.

Index

List of Symbols

Here we list the symbols and abbreviations used in the book, giving a short description of their meaning and, whenever necessary, the number of the page where they are introduced.

$\cong$	The function on the left has the power series on the right as its asymptotic expansion	(p. 65)
$\cong_s$	The function on the left has the power series on the right as its Gevrey expansion of order s	(p. 70)
$\prec$	We write $m \prec n$ whenever, after a rotation of $\mathbb{C}_d$ making the cuts point upward, the nth cut is to the right of the mth one	(p. 148)
$\int_a^{\infty(\tau)}$	An integral from a to infinity along the ray $\arg(u-a)=\tau$	(pp. 78, 79)
du^k	Short for $k\,u^{k-1}du$	(p. 78)
δ	Short for $z(d/dz)$	(p. 24)
$\gamma_k(\tau)$	The path of integration following the negatively oriented boundary of a sector of finite radius, opening larger than π/k and bisecting direction τ	(p. 80)
$(\alpha)_n$	Pochhammer's symbol	(p. 22)

$\mathcal{A}_{\tilde{k},k}$	Ecalle's acceleration operator	(p. 176)
$\hat{\mathcal{A}}_{\tilde{k},k}$	Ecalle's formal acceleration operator	(p. 176)
$\boldsymbol{A}^{(k)}(S,\mathbb{E})$	The space of all $\mathbb{E}$-valued functions that are holomorphic, bounded at the origin and of exponential growth at most k in a sector S of infinite radius	(p. 62)
$\boldsymbol{A}(G,\mathbb{E})$	The space of all $\mathbb{E}$-valued functions that are holomorphic in a sectorial region G and have an asymptotic expansion at the origin	(p. 67)
$\boldsymbol{A}_s(G,\mathbb{E})$	The space of all $\mathbb{E}$-valued functions that are holomorphic in a sectorial region G and have an asymptotic expansion of Gevrey order s	(p. 71)
$\boldsymbol{A}_{s,m}(G,\mathbb{E})$	The space of all $\mathbb{E}$-valued functions that are meromorphic in a sectorial region G and have a Laurent series as asymptotic expansion of Gevrey order s	(p. 73)
$\boldsymbol{A}_s^{(k)}(S,\mathbb{E})$	The intersection of $\boldsymbol{A}_s(G,\mathbb{E})$ and $\boldsymbol{A}^{(k)}(S,\mathbb{E})$	(p. 79)
$\boldsymbol{A}_{s,0}(G,\mathbb{E})$	The set of $\psi \in \boldsymbol{A}_s(G,\mathbb{E})$ with $J(\psi) = \hat{0}$	(p. 116)
$\mathcal{B}_k$	The Borel operator of order k	(p. 80)
$\hat{\mathcal{B}}_k$	The formal Borel operator of order k	(p. 80)
$\mathbb{C}$	The field of complex numbers	
$\mathbb{C}^\nu$	The Banach space of column vectors of length ν with complex entries	(p. 2)
$\mathbb{C}^{\nu\times\nu}$	The Banach algebra of $\nu\times\nu$ matrices with complex entries	(p. 3)
$\mathbb{C}_d$	A complex plane with finitely many cuts along rays $\arg(u-u_m) = -r\,d$	(p. 145)
$C_\alpha(z)$	The kernel of Ecalle's acceleration operator	(p. 175)
$\mathcal{CH}_a$	The Cauchy-Heine operator	(p. 116)
$\widehat{\mathcal{CH}}_a$	The formal Cauchy-Heine operator	(p. 117)
$D(z_0,\rho)$	The disc with midpoint z_0 and radius ρ	(p. 2)
$\deg \hat{T}(z)$	*Degree* or *valuation* of a matrix power series in z^{-1}	(p. 40)
$\mathbb{E}$, $\mathbb{F}$	Banach spaces, resp. Banach algebras	(p. 219)

$\mathbb{E}^*$	The set of continuous linear maps from $\mathbb{E}$ into $\mathbb{C}$	(p. 219)
$\mathbb{E}[[z]]$	The space of formal power series whose coefficients are in $\mathbb{E}$	(p. 64)
$\mathbb{E}[[z]]_s$	The space of formal power series with coefficients in $\mathbb{E}$ and Gevrey order s	(p. 64)
$\mathbb{E}\{z\}$	The space of convergent power series whose coefficients are in $\mathbb{E}$	(p. 64)
$\mathbb{E}\{z\}_{k,d}$	The space of power series with coefficients in $\mathbb{E}$ that are k-summable in direction d	(p. 102)
$\mathbb{E}\{z\}_k$	The space of power series with coefficients in $\mathbb{E}$ that are k-summable in all but finitely many directions	(p. 105)
$\mathbb{E}\{z\}_{T,d}$	The space of power series with coefficients in $\mathbb{E}$ that are T-summable in direction d	(p. 108)
$\mathbb{E}\{z\}_{\boldsymbol{T},d}$	The space of power series with coefficients in $\mathbb{E}$ that are $\boldsymbol{T}$-summable in the multidirection d	(p. 161)
e	Euler's constant (= exp[1])	
$\hat{e}, \hat{0}$	The formal power series whose constant term is e, resp. 0, while the other coefficients are equal to 0	(pp. 64, 70)
$e_1 * e_2$	The convolution of kernel functions	(p. 160)
$E_\alpha(z)$	*Mittag-Leffler's* function	(p. 233)
$F(\alpha;\beta;z)$	Confluent hypergeometric function	(p. 22)
$F(\alpha,\beta;\gamma;z)$	Hypergeometric function	(p. 26)
	Similar notation is used for the generalized confluent hypergeometric function	(p. 23)
	resp. generalized hypergeometric function	(p. 26)
	resp. generalized hypergeometric series	(p. 107)
FFS	Short for *formal fundamental solution*	(p. 131)
$f *_k g$	Convolution of functions f and g	(p. 178)
$\hat{f} *_k \hat{g}$	Convolution of formal power series $\hat{f}$ and $\hat{g}$	(p. 178)
G	A region in the complex domain, resp, on the Riemann surface of the logarithm	(p. 2)
$G(d,\alpha)$	A sectorial region with bisecting direction d and opening α	(p. 61)

$\boldsymbol{H}(G, \mathbb{E})$	The space of functions, holomorphic in G, with values in $\mathbb{E}$ (p. 221)
HLFFS	Short for *highest-level formal fundamental solution* (p. 55)
HLNS	Short for *highest-level normal solution* (p. 138)
J	The linear map that maps functions to their asymptotic expansion (p. 67)
$J_\mu(z)$	Bessel's function (p. 23)
j_0	Number of singular directions in a half-open interval of length 2π (p. 137)
j_1	Number of singular directions in a half-open interval of length $\mu\pi/(qr-p)$ (p. 137)
$\mathcal{L}_k$	The Laplace operator of order k (p. 78)
$\hat{\mathcal{L}}_k$	The formal Laplace operator of order k (p. 79)
$\mathcal{L}(\mathbb{E}, \mathbb{F})$	The Banach algebra of bounded linear maps from $\mathbb{E}$ into $\mathbb{F}$ (p. 219)
$\mathbb{N}$	The set of natural numbers; observe that we here assume $0 \notin \mathbb{N}$
$\mathbb{N}_0$	Is equal to $\mathbb{N} \cup \{0\}$
ODE	Short for *ordinary differential equation*
p.	Short for *page*
PDE	Short for *partial differential equation*
resp.	Short for *respectively*
$\mathbb{R}$	The field of real numbers
$R(z_0, \rho)$	The set of z with $0 < \|z\| < \rho$ (p. 8)
$R(\infty, \rho)$	The set of z with $\|z\| > \rho$ (p. 14)
$r_f(z, N)$	The residue term of order N in the asymptotic expansion of the function f (p. 65)
s_α	The substitution map (p. 78)
$S(d, \alpha)$	The sector with bisecting direction d, opening α and infinite radius (p. 60)

$S(d, \alpha, \rho)$	A sector with bisecting direction d, opening α and radius ρ that may be finite or not	(p. 60)
$\bar{S}(d, \alpha, \rho)$	A closed sector with bisecting direction d, opening α and radius ρ	(p. 60)
$S_\pm$	Two sectors with bisecting direction 0, resp, π, and openings adding up to 2π	(p. 84)
$\mathbf{S}$	Denotes a general summation method	(p. 97)
$\mathbf{S}_A$	Denotes a matrix summation method	(p. 97)
$\mathcal{S}_{k,d}\,(\hat{f})$	The k-sum of $\hat{f}$ in direction d	(p. 100)
$\mathcal{S}_{T,d}\,\hat{f}$	The T-sum of $\hat{f}$ in direction d	(p. 108)
$\mathcal{S}_{\boldsymbol{T},d}\,\hat{f}$	The $\boldsymbol{T}$-sum of $\hat{f}$ in the multidirection d	(p. 161)
Supp_j	A set of pairs (n, m), for which the jth Stokes multipliers may have nontrivial blocks	(p. 141)
$\boldsymbol{T}$	A tuple of integral operators	(p. 161)
$T_1 * T_2$	The convolution of integral operators	(p. 160)
$\mathbb{Z}$	The set of integer numbers	

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